p. 429, last sentence. Deduce that

$$\frac{1}{2\pi i}\oint_L \frac{f(z)}{z-a} = 2\text{ f(a)}$$

Refer to (1), p. 427, which is Cauchy's formula on a simple loop, where f(z) is analytic on and inside L, and a is a point inside L:

$$\frac{1}{2\pi i}\oint_L \frac{f(z)}{z-a} = \text{f(a)} \qquad (1)$$

If an integral of an analytic function g(z) equals f(a) on a simple loop, as in (1), where g(z) = $\frac{f(z)}{z-a}$, then the integral of g(z) on two congruent loops (a small circle traversed 2 times) must equal 2 f(z).

We can also deduce it directly from the statement of Cauchy's Integral Formula in Ch. 8, Ex. 24, on p. 425: *If f is analytic inside K [a closed contour], then*

$$\oint_K \frac{f(z)}{z-a}\,dz = 2\,\pi\,i\,\nu f(z)$$

p. 431, *find the image $\tilde{\xi}$ under $\tilde{z} \mapsto z = f'(z)$ of a short vector emanating from a. Ignoring a term proportional to ξ^2, we find [exercise] that*

$$\tilde{\xi} \equiv f'(\text{a} + \xi) - \text{f'(a)} = \left[\frac{2}{2\pi i}\oint_L \frac{f(z)}{[z-(a+\xi)]^2\,(z-a)}\,\text{dz}\right]\xi$$

We can write

$$\text{f''(a)} = \frac{f'(a+\xi)-f'(a)}{\xi}$$

We are given in (4) that

$$\text{f'(a)} = \frac{1}{2\pi i}\oint_L \frac{f(z)}{(z-a)^2}\text{dz}$$

We will need the expansion of f(z) $(z-[a+\xi])^2$:

$$\text{f(z)}\,(z-[a+\xi])^2 = \text{f(z)}\left(z^2 - 2[\text{a} + \xi]\text{z} + \left[a^2 + 2\text{a}\xi + \xi^2\right]\right)$$

$$\text{f(z)}\,(z^2 - 2\text{az} + a^2) - \text{f(z)}\,2\,\xi\,\text{z} + \text{f(z)}\,2\text{a}\xi + \text{f(z)}\,\xi^2$$

$$f(z)\,(z-a)^2 - f(z)\,2\xi\,(z\text{-}a) \quad \text{by grouping and deleting } \xi^2 \qquad (2)$$

Then

$$\tilde{\xi} \equiv f'(a+\xi) - f'(a) = \frac{1}{2\pi i}\left[\oint_L \frac{f(z)}{(z-[a+\xi])^2}dz - \oint_L \frac{f(z)}{(z-a)^2}dz\right]$$

$$= \frac{1}{2\pi i}\oint_L\left[\frac{f(z)}{(z-[a+\xi])^2} - \frac{f(z)}{(z-a)^2}\right] dz$$

$$= \frac{1}{2\pi i}\oint_L\left[\frac{f(z)\,(z-a)^2 - f(z)\,(z-[a+\xi])^2}{(z-[a+\xi])^2\,(z-a)^2}\right] dz$$

$$= \frac{1}{2\pi i}\oint_L\left[\frac{f(z)\,(z-a)^2 - \left(f(z)\,(z-a)^2 - f(z)\,2\,\xi\,(z-a)\right)}{(z-[a+\xi])^2\,(z-a)^2}\right] dz \quad \text{by substitution from (2)}$$

$$= \frac{1}{2\pi i}\oint_L \frac{2\,\xi\,f(z)}{(z-[a+\xi])^2\,(z-a)}\,dz \qquad \text{add like terms, div. top and bottom by (z-a)}$$

$$= \left[\frac{2}{2\pi i}\oint_L \frac{f(z)}{(z-[a+\xi])^2\,(z-a)}dz\right]\xi \qquad \text{move constants outside integral}$$

p. 433. Verify the two equations in the middle of the page and the limit $\lim_{N\to\infty} f_N(z) = f(z)$. This problem was not posed by Needham, but I found the page to be heavy slogging, so I wrote up some notes.

$$f(z) - f_N(z) = \frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z}\left(\frac{1}{1-(z/Z)}\right)dZ - \frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z}(z/Z)^{N-1}\,dZ$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z}\left(\frac{1}{1-(z/Z)} - (z/Z)^{N-1}\right) dZ$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z}\left(\frac{1}{1-(z/Z)} - \left[1 + (z/Z) + (z/Z)^2 + \cdots + (z/Z)^{N-1}\right]\right) dZ$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(Z)}{Z}\left(\frac{(z/Z)^N}{1-(z/Z)}\right) dZ$$

Vasco pointed out in the forum that $[1 + (z/Z) + (z/Z)^2 + \cdots + (z/Z)^{N-1}]$ can be written $\frac{1-(z/Z)^N}{1-(z/Z)}$. Then

$$\frac{1}{1-(z/Z)} - \frac{1-(z/Z)^N}{1-(z/Z)} = \frac{(z/Z)^N}{1-(z/Z)}$$

We could also derive it as follows:

$$[1 + (z/Z) + (z/Z)^2 + (z/Z)^3 + \cdots\ ] - [1 + (z/Z) + (z/Z)^2 + (z/Z)^3 + \cdots (z/Z)^{N-1}\ ]$$

$= (z/Z)^N + (z/Z)^{N+1} + (z/Z)^{N+2} + \cdots$

$= (z/Z)^N\ [1 + (z/Z) + (z/Z)^2 + (z/Z)^3 + \cdots\ ]$

$= (z/Z)^N \frac{1}{1-(z/Z)} = \frac{(z/Z)^N}{1-(z/Z)}$

This verifies the equation. Then, from the last equation on p. 432 and from ¶ 1, line 2, p. 433, *it follows that*

$f(z) - f_N(z) = \frac{1}{2\pi i} \oint_C \frac{f(Z)}{Z} [\frac{1}{1-(z/Z)} - \frac{1-(z/Z)^N}{1-(z/Z)}]\ dZ$

$= \frac{1}{2\pi i} \oint_C \frac{f(Z)}{Z} [\frac{(z/Z)^N}{1-(z/Z)}]\ dZ$

$= \frac{1}{2\pi i} \oint_C \frac{(z/Z)^N f(z)}{(Z-z)} dZ$

Then we can establish that $f_N(z)$ *tends to f(z) as N tends to infinity. The modulus of an integral cannot exceed the product of the length of the path and the maximum modulus of the integrand at points on the path.*

$|f(z) - f_N(z)| = |\frac{1}{2\pi i} \oint_C \frac{(z/Z)^N f(z)}{(Z-z)} dZ| \leq \text{Circumference(C)} \cdot \frac{1}{2\pi} \cdot \max [\,|f(z)/(Z-z)| \cdot |z/Z|^N]$

The length of the path is the circumference of C, which is 2πR. But $|z/Z|^N$ shrinks with increasing N, so we need only maximize the modulus of |f (z)/(Z-z)|:

M = max[f (z)/(Z-z)]

Rewrite with substitutions for path length and maximum modulus.

$|f(z) - f_N(z)| = \frac{2\pi R}{2\pi} M\ |(z/Z)|^N = R\ M\ |(z/Z)|^N$

Then $\lim_{N\to\infty} |(z/Z)|^N = 0$, so $\lim_{N\to\infty} |f(z) - f_N(z)| = 0$, and $\lim_{N\to\infty} f_N(z) = f(z)$.

p. 434, Italics are quotes or partial quotes.

$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$

Alternatively [exercise], the existence of this series may be deduced by directly generalizing the argument leading to the origin-centred series.

If f(z) is analytic on and inside a circle of radius R centred on a point a, then f(z) may be expressed as a power series that converges on this disc:

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots.$$

It follows that the coefficients c_n may be expressed as

$$c_n = \frac{f^{(n)}(a)}{n!},$$

so the power series is actually a Taylor series, and the coefficients do not depend on R:

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f^{(3)}(a)}{3!}(z-a)^3 + \cdots.$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}\,(z-a)^2$$

This is what we wanted to deduce. To verify it, we continue as Needham did:

To establish the existence of this series, we return to Cauchy's formula (1):

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(Z)}{(Z-a)-(z-a)}dZ = \frac{1}{2\pi i}\oint_C \frac{f(Z)}{(Z-a)}\left[\frac{1}{1-(z-a)/(Z-a)}\right]dZ$$

Since z is inside the circle on which Z lies, |z-a| < |Z-a| = R, and |(z-a)/(Z-a)| ≤ 1. Thus, $\left[\frac{1}{1-(z-a)/(Z-a)}\right]$ *may be viewed as the sum of an infinite geometric series*, and

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(Z-a)}{(Z-a)}\left[1 + ((z-a)/(Z-a)) + ((z-a)/(Z-a))^2 + ((z-a)/(Z-a))^2 + \cdots\right] dZ$$

Provided it makes sense to integrate this infinite series term by term, we deduce that f(z) can indeed be expressed as a power series:

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z-a)^2, \text{ where } c_n = \frac{1}{2\pi i}\oint_C \frac{f(Z)}{(Z-a)^{n+1}}dZ \qquad (8)$$

Consider the sum $f_N(z) \equiv \sum_{n=0}^{N-1} c_n\,(z-a)^2$ *of the first N terms of the series* (8) [not same as Needham's (8)*]. The result will be established if we can show that* $f_N(z)$ *tends to* $f_N(z)$ *as N tends to infinity*. By our previous reasoning from p. 433,

$$\frac{1}{1-((z-a)/(Z-a))} - \frac{1-((z-a)/(Z-a))^N}{1-((z-a)/(Z-a))} = \frac{((z-a)/(Z-a))^N}{1-((z-a)/(Z-a))}$$

It follows that

$$f(z) - f_N(z) = \frac{1}{2\pi i}\oint_C \frac{((z-a)/(Z-a))^N f(Z)}{(Z-z)}dZ$$

Let M = max |f(Z)/(Z-z)|.

$$| f(z) - f_N(z) | \leq \text{R M } | (z-a)/(Z-a) |^N$$

Thus, $\lim_{n\to\infty} f_N(z) = f(z)$*, as was to be shown.*

p. 438 [exercise] at top of page.

Given that

$$\text{I} = \text{-2i} \oint_C \frac{\text{dz}}{z^2+2\,\text{az}+1} \qquad \text{from p. 437, bottom}$$

$$= \text{-2i}\,(2\pi \text{i Res}\left[\frac{1}{(z-p)(z-q)}, q\right]), \qquad (1)$$

and let f(z) = $\frac{1}{z-p}$/(z-q), P(z) = $\frac{1}{z-p}$, Q(z) = $z - q$

Then

$$\text{Res}\left[\frac{1}{(z-p)(z-q)}, q\right] = \frac{P(q)}{Q'(q)} = \frac{1}{z-p} = \frac{1}{q-p}$$

and substituting into (1),

$$\text{I} = 4\pi \text{ Res}\left[\frac{1}{(z-p)(z-q)}, q\right] = \frac{4\pi}{q-p} \qquad (2)$$

Now [exercise] show that (top p. 438)

$$\text{I} = 4\pi \text{ Res}\left[\frac{1}{(z-p)(z-q)}, q\right] = \frac{4\pi}{(q-p)} = \frac{2\pi}{\sqrt{a^2-1}}$$

$$z^2 + 2\text{ az} + 1 = \text{(z-p)(z-q)}$$

Since p and q are singularities, (z-p)(z-q) = 0 when z = p or z = q, so

$$z^2 + 2\text{ az} + 1 = 0$$

$$\text{z} = \frac{-2a \pm \sqrt{(2a)^2-4}}{2}$$

$$= -a \pm \sqrt{a^2-1}$$

If q = $-a + \sqrt{a^2 - 1}$, then p = $-a - \sqrt{a^2 - 1}$.

Alternatively, since pq = 1,

$$p = \frac{1}{-a+\sqrt{a^2-1}} = \frac{-a-\sqrt{a^2-1}}{\left(-a-\sqrt{a^2-1}\right)\left(-a+\sqrt{a^2-1}\right)}$$

$$= \frac{-a-\sqrt{a^2-1}}{a^2-a^2+1} = -a - \sqrt{a^2-1}$$

then

$$q\text{-}p = -a + \sqrt{a^2-1} - (-a - \sqrt{a^2-1})$$

$$= 2\sqrt{a^2-1}$$

Substituting into (2),

$$I = \frac{4\pi}{q-p} = \frac{4\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$$

Hence, we have shown that

$$I = 4\pi\,\mathrm{Res}\left[\frac{1}{(z-p)(z-q)}, q\right] = \frac{4\pi}{(q-p)} = \frac{2\pi}{\sqrt{a^2-1}}$$

p. 438. Derivation of series for cot πz.

$$\cot \pi z = \frac{\cos \pi z}{\sin \pi z} = \cos(\pi z)\csc(\pi z)$$

Then convert to series and divide cosecant by πz.

p. 438-439, § III, §§ 4

Beginning with ¶ 3, the method is to rewrite g(z) = $\left(1/z^2\right)\cot(\pi z)$ as a Laurent series and find the residue. Since f(z) = $\left(1/z^2\right)$ is non-singular at n and cot(π z) is periodic, then

$$\mathrm{Res}[f(z)\cot(\pi z), n] = f(n)\,\mathrm{Res}[\cot(\pi z), 0]$$

or, where f(z) is non-singular at n, and h(z) is periodic,

$$\mathrm{Res}[f(z)\,h(z), n] = f(n)\,\mathrm{Res}[h(z), 0].$$

Where is this justified in the text? From this equation, we get $\text{Res}\left[1 / z^2 \cot(\pi\, z), n\right] = 1 / \left(\pi n^2\right)$.

p. 439, $\mathbb{S}$ III, $\mathbb{SS}$ 4, ¶ 1. *Note that if f(z) is any analytic function that is non-singular at n, then*

$$\mathit{Res}[f(z)\, \cot(\pi\, z),\, n] = \frac{1}{\pi} f(n)$$

This may also be verified using (11): $\text{Res}[f(z), a] = \frac{P(a)}{Q'(a)}$, where *f(z) has a "simple" (i.e. order 1) pole at* ***a*** *as a result of Q having a simple root at that point.*

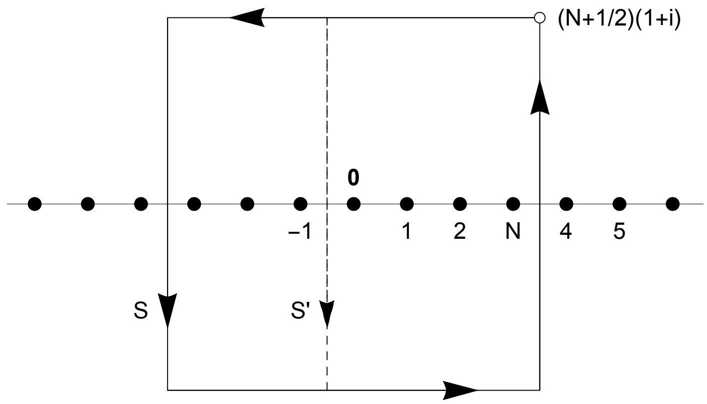
Substitute f(z)cot(π z) for f(z) and **n** for **a** in (11):

$$\text{Res}[f(z)\, \cot(\pi\, z), n] = \text{Res}[\, \frac{f\,(z) \cos\,(\pi\, z)}{\sin(\pi\, z)}, n]$$

We know that f(z) cot(π z) has a simple pole at **n** because sin has a Taylor series with a simple root at **n**. And of course, $\sin(\pi n) = 0$. Then

$$\text{Res}[f(z)\, \cot(\pi\, z), n] = \frac{f\,(n) \cos\,(\pi\, z)}{(\sin(\pi\, z))'}$$

$$= \frac{f\,(n)\; \cos(\pi\, z)}{\pi\, \cos(\pi\, z)} \;\; = \frac{1}{\pi}\, f\,(n)$$



from Figure [5], p. 440

p. 439

Needham established that

$$\text{Res}\left[\frac{1}{z^2}\cot(\pi z),\ 0\right] = -\frac{\pi}{3} \text{ (corresponding to pole of order 2)} \qquad (1)$$

$$\text{Res}\left[\frac{1}{z^2}\cot(\pi z),\ n\right] = \frac{1}{\pi n^2} \text{ (corresponding to singularities at n)} \qquad (2)$$

The underlying generalization here seems to be that Res[f(z) h(z), n] = f(n) Res[h(z), 0], where n is non-singular at n and h(z) is periodic. Where n is singular, we have to find the Laurent series and the residue of 1/z. We can use Res[h(z), 0] where h(z) is periodic and Res[h(z), n] = Res[h(z), 0].

Euler used a different method when he developed the proof of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, where n is a positive integer. Whoever developed the proof in our text must have known the values in (1) and (2). For this proof, one needs terms of known value in an equation with $1 / n^2$. We see from this page that the residue $\text{Res}\left[\frac{1}{z^2}\cot(\pi z),\ n\right]$ contains $1 / n^2$. A contour integral based on residues could provide the known quantities for the equation: the value of the contour integral and the value of other residues. Needham provided a suitable contour integral based on the function g(z) = $\frac{1}{z^2}\cot(\pi z)$ applied to the square closed contour in Figure [5].

How did the developer of this proof know to use the whole square? He might have used just over half of the square by drawing the left side at x = -1/2 (dotted line, Figure [5], S'). With this arrangement, as N increases, we no longer have a square. We have a rectangle that expands horizontally to the right and vertically in both directions while maintaining a side of expanding length at -(1/2). The rectangle encircles the origin and includes N poles in the positive real integers. We need to show that $|\oint_{S'} g(z)dz| < \text{Max}|(1/z^2)|$ (length of perimeter) $\longrightarrow 0$. This amounts to showing that |z| > N for all z on S'. On the left vertical side, we can only claim that |z| ≥ 1/2. Thus, we have not met the criterion that |z| > N throughout S', so we can't say that $|1/z^2| < 1/N^2$. Hence, we can't claim that $|\oint_{S'} g(z)dz| \longrightarrow 0$. S as drawn and constructed permits us to claim |z| > N for all z in S.

p. 440, *We begin with the horizontal edges, $y = \pm\left(N + \frac{1}{2}\right)$. Since* $|e^{\pm i\pi z}| = e^{\mp\pi y}$ [demonstrated below]*, it is not hard to see [exercise] that if N is reasonably large then |cot(πz)| is very close to 1.*

Show $|e^{\pm i\pi z}| = e^{\mp\pi y}$

$$|e^{\pm i\pi z}| = |e^{\pm i\pi(x+iy)}| = |e^{\pm(i\pi x)}\, e^{\mp\pi y}| = e^{\mp\pi y}$$

Then

$$|\cot(\pi z)| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| = \left|\left(\frac{e^{i\pi z}}{e^{i\pi z}}\right)\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| = \left|\frac{e^{2 i\pi z} + 1}{e^{2 i\pi z} - 1}\right|$$

If $|e^{2 i\pi z}|$ is very small, then |cot(πz)| is very close to 1. Since $|e^{2 i\pi z}| = e^{-2\pi y}$, and $y = \pm\left(N + \frac{1}{2}\right)$, then $|e^{2 i\pi z}| = e^{-2\pi y}$ becomes very small for increasing positive N ("N is reasonably large"), and |cot(πz)| is very close to 1.

Finally, on the vertical edges we have $z = \pm(N + \frac{1}{2}) + iy$, and it follows [exercise] that

$$|\cot(\pi z)| = \left|\frac{1 - e^{-2\pi y}}{1 + e^{-2\pi y}}\right| \le 1$$

Multiply top and bottom by $e^{-i\pi z}$

$$|\cot(\pi z)| = \left|\frac{e^{i\pi z - i\pi z} + e^{-i\pi z - i\pi z}}{e^{i\pi z - i\pi z} - e^{-i\pi z - i\pi z}}\right| = \left|\frac{1 + e^{-2 i\pi z}}{1 - e^{-2 i\pi z}}\right|$$

The numerator:

$$\left|1 + e^{-2 i\pi\left[\pm\left(N+\frac{1}{2}\right) + i y\right]}\right| = |1 + e^{\mp 2\pi i N \mp i\pi + 2\pi y}|$$

$$= |1 - e^{\mp 2\pi i N + 2\pi y}| = |1 - e^{2\pi y}|$$

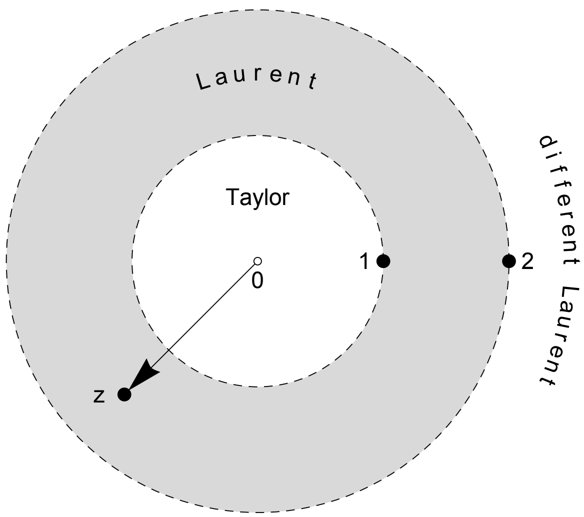
The denominator

$$|1 - e^{-2\,i\pi z}| = \left|1 - e^{-2\,i\pi\left[\pm\left(N+\frac{1}{2}\right)+i\,y\right]}\right|$$

$$= \left|1 - e^{\mp 2\,\pi i\,N \mp i\pi + 2\,\pi\,y}\right| = |1 + e^{2\,\pi\,y}|$$

Then

$$|\cot(\pi z)| = \frac{|1-e^{2\,\pi\,y}|}{|1+e^{2\,\pi\,y}|} = \frac{e^{-2\,\pi\,y}|1-e^{2\,\pi\,y}|}{e^{-2\,\pi\,y}|1+e^{2\,\pi\,y}|} = \frac{|e^{-2\,\pi\,y}-1|}{|e^{-2\,\pi\,y}+1|} = \frac{|1-e^{-2\,\pi\,y}|}{|1+e^{-2\,\pi\,y}|} \leq 1$$



from Figure [6a], p. 443

p. 442, § IV, §§ 1 An Example, ¶ -1. *Finally, in the region |z| > 2 beyond the annulus we obtain [exercise] a different Laurent series:*

$$F(z) = \frac{1}{z^2} + \frac{3}{z^3} + \cdots + \frac{\left(2^{n-1}-1\right)}{z^n} + \cdots, \quad \text{for } |z| > 2.$$

In the partial fraction F(z), in order to express the change in F(z) as z approaches the radius of the singularity, we need a term with a denominator containing (1-A), where A < 1 and A = const./z or A = z/const. In this case, |z| > 2, so if A < 1, we write (1-2/z). Than as z shrinks near 2, F(z) expands rapidly. If it were the case that |z| < 2, as inside the annulus, we would write (1 - z/2), as Needham did.

Once the (1-A) term has been decided upon, the corresponding term of the partial fraction is rewritten to display this term. Instead of being displayed in the form (2-z), the singularity at 2 is now displayed in the form z(1-2/z). The change in sign of the partial fraction is an artifact of the rewrite.

$$-\frac{1}{(2-z)} = -\frac{1}{z\left(\frac{2}{z}-1\right)} = \frac{1}{z\left(1-\frac{2}{z}\right)}$$

When z is very large, the partial fraction goes to 0. As |z| shrinks to 2, this term goes to ∞. Since (2/z) < 1, we can write the term as the series

$$F(z) = \frac{1}{z}[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \left(\frac{2}{z}\right)^4 + \cdots] = \sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \cdots \qquad (1)$$

Each of the regions (disc, annulus, beyond the annulus) can be described by the sum of two terms that can each be written as convergent series. The two terms are simply rewritten versions of the original partial fraction F(z). The series must both converge within the region of interest. The region |z| > 2 outside the shaded annulus (last sentence, § 1) can be described by two series that converge in the region |z| > 2 (but only z's from the region |z| > 2 lie in the domain of this rewritten F(z)). A series that converges in the region |z| > 1 also converges in the region |z| > 2, so the two series can be added to obtain a Laurent series for the region |z| > 2. For the region |z| > 1, we have (-(1/z[1-(1/z)]) = $[-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4} - \cdots] = -\sum_{n=1}^{\infty}(1/z)^{n+1}$ for the region outside the inner boundary of the annulus. To this we add (1) for the region beyond the annulus.

$$\begin{array}{ll} & \quad\text{for } |z| > 1 \qquad\qquad\qquad \text{for } |z| > 2 \\ F(z) = & \text{Series}[-1/z[1 - (1/z)]] + \text{Series}[\frac{1}{z[1-(2/z)]}] \end{array}$$

$$= -\sum_{n=1}^{\infty}(1/z)^{n+1} \qquad + \qquad \frac{1}{z}\sum_{n=0}^{\infty}(2/z)^n$$

$$= [-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4} - \cdots] \quad + \quad [\frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{8}{z^4} + \cdots]$$

$$= [\frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \cdots + \frac{(2^{n-1}-1)}{z^n} + \cdots]$$

$$= \sum_{n=1}^{\infty}\frac{(2^{n-1}-1)}{z^n}$$

p. 445, *Essentially identical reasoning [exercise] also justifies term-by-term integration in the case of the integral round $\mathcal{D}$.*

Since $\frac{W}{z} < 1$, we can write [1-$\frac{W}{z}$] as a geometric series:

$$\frac{1}{2\pi i}\oint_{\mathcal{D}} \frac{f(W)}{z}\,\frac{1}{\left[1-\frac{W}{z}\right]}\,\mathrm{dW} = \frac{1}{2\pi i}\oint_{\mathcal{D}} \frac{f(W)}{z}\left[1+\frac{W}{z} + \left(\frac{W}{z}\right)^2 + \left(\frac{W}{z}\right)^3 + \cdots\right]\mathrm{dW}$$

$$= \frac{1}{2\pi i}\oint_{\mathcal{D}} f(W)\left[\frac{1}{z}+\frac{W}{z^2} + \frac{W^2}{z^3} + \frac{W^3}{z^4} + \cdots\right]\mathrm{dW}$$

$$= \sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\oint_{\mathcal{D}} W^{n-1} f(W)\,\mathrm{dW}\left(\frac{1}{z}\right)^n\right],$$ with brackets rearranged to include $\left(\frac{1}{z}\right)^n$.