Needham, Visual Complex Analysis, Chapter 9, Exercise 9, p. 448
G. Palmer, December 3, 2017, 2:40 pm PT

9. Continuing from the previous question, consider the wedge-shaped contour K shown below.

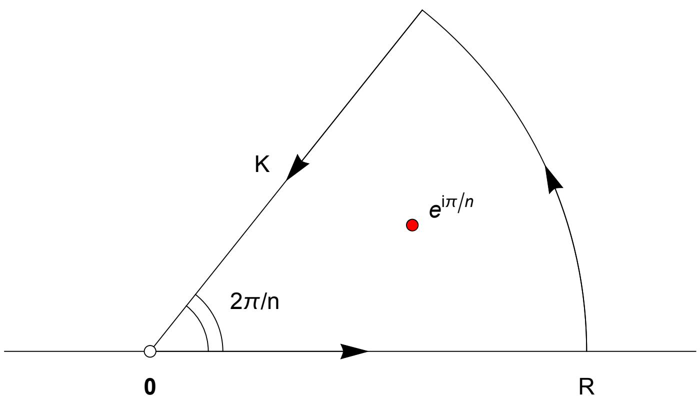

Figure 1. Contour integral of wedge with pole
(Needham, p. 448)

(i) Use the Residue Theorem to show that if n = 2, 3, 4, ..., and R > 1 (as illustrated), then

$$\oint_K \frac{dz}{1+z^n} = -\frac{2\pi i}{n}\, e^{i(\pi/n)}$$

R must be greater than 1 in order to circle a pole of (negative unity). We can use the Residue Theorem plus our understanding that $F_n(z)$ on the wedge has one pole at $e^{i\pi/n}$ to calculate the residue by the same method as in 8 (i, ii). The derivation of residues does not depend on even n:

$$\oint_K \frac{\mathrm{dz}}{1+z^n} = 2\pi \mathrm{i}\ \mathrm{Res}[\mathrm{f(z)}, e^{\mathrm{i}\pi/n}] = 2\pi\mathrm{i}(\text{-p/n}) = 2\pi\mathrm{i}\cdot(-e^{i(\pi/n)}/\mathrm{n}) = -\frac{2\pi i}{n}\, e^{i(\pi/n)}$$

(ii) Show that

$$\underset{R\longrightarrow\infty}{Lim} \oint_K \frac{dz}{1+z^n} = \left[1 - e^{i(2\pi/n)}\right]\int_0^\infty \frac{dx}{1+x^n}.$$

Integrate the three parts of the wedge separately: segment on real line, arc, ray at arg = 2π/n. We know the value of the segment on the real line from exercise 8 (iii), but it won't be needed for this proof. We will show that $\oint_{\mathrm{arc}} \frac{\mathrm{dz}}{1+z^n}$ vanishes as R goes to infinity. The integral of F_n(z) on the ray, which is traversed from $e^{i(2\pi/n)}$ to 0, can be parameterized on x.

$$\underset{R\longrightarrow\infty}{\text{Lim}} \oint_K \frac{dz}{1+z^n} = \int_0^\infty \frac{dx}{1+x^n} + \oint_{\text{arc}} \frac{dz}{1+z^n} - e^{i(2\pi/n)} \int_0^\infty \frac{dx}{1+x^n}$$

real seg. arc ray

Show that $\oint_{\text{arc}} \frac{dz}{1+z^n}$ vanishes using inequality (5) from p. 387. The minimum absolute value of the denominator is $|(1 - R^n)|$ for odd n, so M is the reciprocal $1/(1-R^n)$.

$$\left|\frac{1}{1+z^n}\right| \le \text{M } (2\pi/\text{n}) \text{ R} = \frac{2\pi\text{R}}{n|(1-R^n)|}$$

which approaches $\frac{1}{R}$ and vanishes in the limit. Then the contour integral for the arc must also vanish.

Show that the integral on the ray = $\int_0^\infty \frac{-e^{i(2\pi/n)}\,dx}{1+x^n}$ by parameterizing $\oint_K \frac{dz}{1+z^n}$ on x. The direction of z on the ray is towards the origin.

$$z = e^{i2\pi/n}x, \qquad dz = e^{i2\pi/n}dx$$

$$\oint_{\text{ray}} \frac{dz}{1+z^n} = \int_0^\infty \frac{e^{i(2\pi/n)}\,dx}{1+\left(e^{i2\pi/n}\,x\right)^n} = e^{i(2\pi/n)} \int_0^\infty \frac{dx}{1+x^n}$$

Adding the results for the three segments, we can write:

$$\underset{R\longrightarrow\infty}{\text{Lim}} \oint_K \frac{dz}{1+z^n} = \int_0^\infty \frac{dx}{1+x^n} + 0 - e^{i(2\pi/n)} \int_0^\infty \frac{dx}{1+x^n} = [1 - e^{i(2\pi/n)}] \int_0^\infty \frac{dx}{1+x^n}$$

(iii) Deduce that (15) is indeed valid for odd n as well as even n.

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n\sin(\pi/n)} \qquad (15)$$

This is an attempt to reproduce Vasco’s proof. If (15) is valid for odd n as well as even n, then, since we know that the Residue Theorem is valid for all *n = 2, 3, 4,* and R > 1,

$$[1 - e^{i(2\pi/n)}] \int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}\, e^{i(\pi/n)} \quad \text{from (i)}$$

$$e^{-i\pi/n}[1 - e^{i(2\pi/n)}] \int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}\, e^{i(\pi/n)} e^{-i\pi/n}$$

$$[e^{-i\pi/n} - e^{i(\pi/n)}] \int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n}$$

$$\int_0^\infty \frac{dx}{1+x^n} = -\frac{2\pi i}{n\left[e^{-i\pi/n} - e^{i(\pi/n)}\right]} = -\frac{2\pi i}{\text{n}2\,i\,(-\sin(\pi/n))} = \frac{\pi}{n\,sin(\pi/n)}$$

Therefore, (15) is valid for both odd and even n.