Needham, Visual Complex Analysis, Chapter 9, Exercise 8, p. 447-8
G. Palmer, November 30, 2017, 11:30 pm PT

8 Let $F_n(z) = 1 / (1 + z^n)$, where n is even.

(i) Use (11) to show that if p is a pole of F_n the Res$[F_n, p] = -(p/n)$.

Since p is a pole of F_n, $p^n = -1$. Let P = 1, let Q = $(1 + z^n)$.

$$\text{Res}[F_n, \text{p}] = \text{P/Q'} = \frac{1}{\text{nz}^{n-1}}|_p = \frac{z}{\text{nz}^n}|_p = \frac{p}{n(-1)} = -\frac{p}{n}$$

(ii) With the help of part (i) show that the residues of F_n in the upper half plane is a geometric series with sum $1/[i\ n \sin(\pi/n)]$.

Since p is a pole of F_n, it is a root of $(1 + z^n)$. Then p = $\sqrt[n]{-1}$. Since n is even, the factors of the polynomial come in conjugate pairs. Half the roots lie above the real line and half lie below, so there must be n/2 roots above and n/2 roots below. The first root must be at $e^{i\pi/n}$, n > 1. By the roots of unity (pp. 26-27), the roots of $(1 + z^n)$ are $e^{i(\pi/n+k2\pi/n)}$, where k = 0..(n-1).

Then, the poles and residues for n = 2, 4, 6 are as in Tables 1 and 2.

Table 1. Roots of unity for $(1 + z^n)$ (same as poles of F_n; roots in UHP in boldface)

n	
2	$\{e^{i\pi/2}, e^{i(\pi/2+\pi)}\} = \{\mathbf{i}, -i\}$
4	$\{e^{i\pi/4}, e^{i(\pi/4+\pi/2)}, e^{i(\pi/4+\pi)}, e^{i(\pi/4+3\pi/2)}\} = \{\boldsymbol{e^{i\pi/4}}, \boldsymbol{e^{i3\pi/4}}, e^{i5\pi/4}, e^{i7\pi/4}\}$
6	$\{\boldsymbol{e^{i\pi/6}}, \boldsymbol{i}, \boldsymbol{e^{i5\pi/6}}, e^{i7\pi/6}, -i, e^{i11\pi/6}\}$

Table 2. Poles in UHP, residues (-p/n), and sums compared to $1/[i\ n \sin(\pi/n)]$

n	**p in UHP**	**-p/n**	**Sum(-p/n)**	**1/[i n sin(π/n)]**
2	i	-i/2	-i/2	-i/2
4	$e^{i\pi/4}, e^{i3\pi/4}$	$-e^{i\pi/4}/4, -e^{i3\pi/4}/4$	$-i/2\sqrt{2}$	$-i/2\sqrt{2}$
6	$e^{i\pi/6}, i, e^{i5\pi/6}$	$-e^{i\pi/6}/6, -i/6, -e^{i5\pi/6}/6$	-i/3	-i/3

In a first attempt, the numbers in the last two columns of Table 2s were equal for every F_n but for opposite signs. I concluded that the phrase "the residues of F_n in the upper half plane" refers to F_n when **p** is in the UHP, rather than to the residues themselves in the UHP. It seems that z is the default figure in reference to locations.

In seeking geometric progressions in the residues, we find that for n = 2, no ratio can be calculated for the residues in the UHP, but if we include poles for the LHP the ratio would be $-1 = e^{i\pi}$. For n = 4, the ratio is $i = e^{i\pi/2}$. For n = 6, the ratio is $e^{i\pi/3}$. Hence, it appears that for each succeeding n, the ratio is $e^{i2\pi/n}$. We can verify this:

$$e^{i(\pi/n\,+\,(k+1)\,2\,\pi/n)}/e^{i(\pi/n\,+\,\mathrm{k}2\pi/n)} = e^{\mathrm{i}2\pi/n}$$

So the residues of each F_n fit a geometric progression with a predictable ratio.

Show that the sum of the residues of F_n in the UHP equals 1/[i n sin(π/n)]. For a formal proof, one would have to show that

$$\sum_{k=0}^{n/2} (-p_k\,/\,n) = 1/[\mathrm{i\ n}\sin(\pi/\mathrm{n})], \text{ for } p_k \text{ in UHP}$$

The formula for the sum of a series is

$$S_n = \frac{a-\mathrm{ar}^n}{1-r},$$

where a is the first term of the series and r is the common factor (Kline, M. *Calculus*, 1998), so we can write

$$\sum_{k=0}^{n/2} (-p_k\,/\,n) = 1/[\mathrm{i\ n}\sin(\pi/\mathrm{n})]$$

$$-\frac{1}{n}\sum_{k=0}^{n/2-1} p_k = -\frac{1}{n}e^{i(\pi/n)}\sum_{k=0}^{n/2-1} e^{\mathrm{k}2\pi/n}$$

$$= -\frac{1}{n}e^{i(\pi/n)}\,[1 + e^{2\,\pi/n} + e^{4\,\pi/n} + e^{6\,\pi/n} + \cdots]$$

$$= -\frac{1}{n}e^{i(\pi/n)}\,\frac{1-\left(e^{\mathrm{i}2\pi/n}\right)^{n/2}}{1-e^{\mathrm{i}2\pi/n}} = -\frac{1}{n}e^{i(\pi/n)}\,\frac{1-e^{\mathrm{i}\pi}}{1-e^{\mathrm{i}2\pi/n}} = -\frac{1}{n}e^{i(\pi/n)}\,\frac{e^{-i(\pi/n)}}{e^{-i(\pi/n)}}\,\frac{2}{1-e^{\mathrm{i}2\pi/n}}$$

$$= -\frac{1}{n}\,\frac{2}{e^{-i(\pi/n)}-e^{\mathrm{i}\pi/n}} = \frac{1}{n}\,\frac{2}{e^{i(\pi/n)}-e^{-\mathrm{i}\pi/n}} = \frac{2}{n\,(2\,i\,)\sin(\pi/n)} = \frac{1}{n\,i\sin(\pi/n)}$$

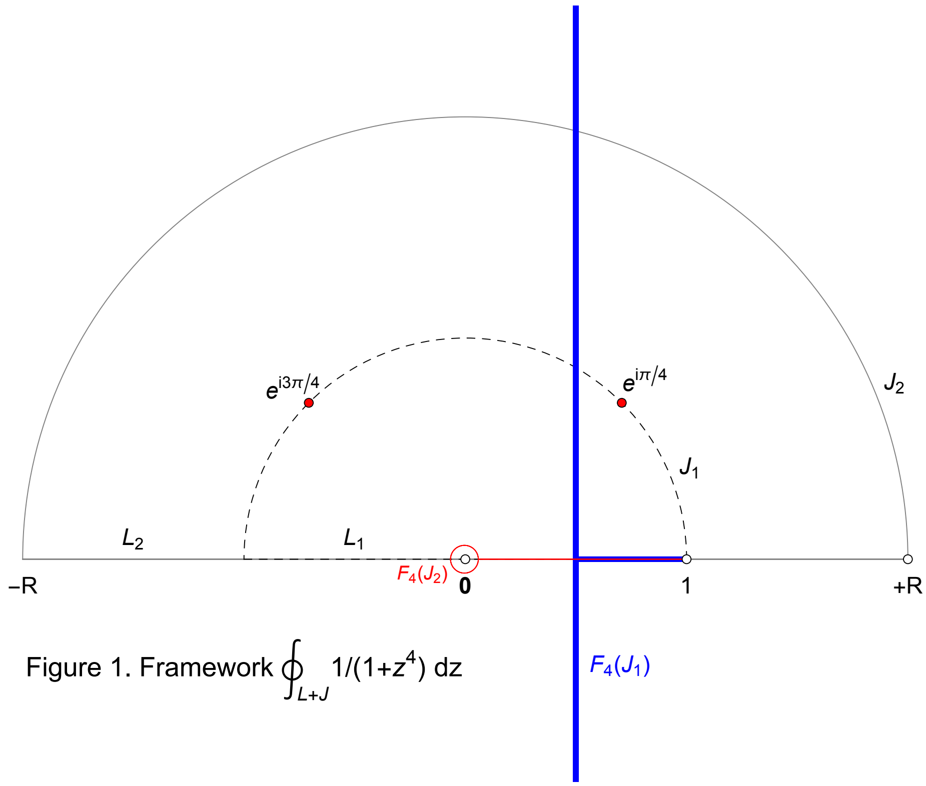

Figure 1. Framework $\oint_{L+J} 1/(1+z^4)\, dz$

(iii) By applying the Residue Theorem in the contour L+J, shown in [4a], deduce that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi}{n \sin(\pi/n)} \qquad (15)$$

Figure 1 is similar to [4a] but adapted to this problem. It shows $F_n(\text{z})$ in blue and red on two closed half circles $(L_1 + J_1)$ and $(L_2 + J_2)$ respectively with n = 4, and it shows the two poles of the UHP in red. The small red circle around the origin and the thick vertical blue line are as labeled: $F_4(J_1)$ and $F_4(J_2)$. The remaining colored parts on the x axis are $F_4(L_1)$ and $F_4(L_2)$. No point is being made with these, but it can be useful to see the shape of the curve when integrating.

Based on (i) and (ii), we can write an integral for F_n on L+J.

$$\oint_{L+J} \frac{\text{dz}}{1+z^n} = 2\,\pi \text{i} \sum_{\text{UHP}} \text{Res}[F_n(\text{z}),\, p]_{\text{UHP}} = 2\pi \text{i}\, \frac{1}{i\, n \sin(\pi/n)} = \frac{2\,\pi}{n \sin(\pi/n)}$$

From p. 236, bottom, we can write

$$\oint_{L+J} \frac{\text{dz}}{1+z^n} = \int_{-R}^{+R} \frac{1}{1+x^2}\, dx + \int_J \frac{1}{1+z^n}\, dz$$

Since we are allowing R to go to ∞, we can write

$$= \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx + \int_J \frac{1}{1+z^n}\, dz = \frac{2\,\pi}{n \sin(\pi/n)}$$

Then it appears that we can get the correct answer if we can show that $\int_J \frac{1}{1+z^n}\, dz$ vanishes as R $\longrightarrow \infty$. From (5), p. 387, we can write

$$\underset{R\longrightarrow\infty}{\text{Lim}} \left|\int_J \frac{1}{1+z^n}\, dz\right| \le \text{M (length of J)} = \underset{R\longrightarrow\infty}{\text{Lim}}\left[\left|\frac{1}{1+\text{Re}^{in\theta}}\right| \pi\, \text{R}\right]$$

Note that the minimum value of $\left|1+\text{Re}^{in\theta}\right|$, occurs where $\theta = \pi$, because this is where z is the negative real number -R. At this point, this minimum absolute value can be written R^n-1. The maximum value |f(z)| is the reciprocal of that and therefore, M = $1/(R^n - 1)$.

$$\underset{R\longrightarrow\infty}{\text{Lim}} \left|\int_J \frac{1}{1+z^n}\, dz\right| \le \underset{R\longrightarrow\infty}{\text{Lim}}\left[\frac{1}{R^n-1}\,\pi\, \text{R}\right] = 0$$

This shows that $\int_J \frac{1}{1+z^n}\, dz$ vanishes as |z| $\longrightarrow \infty$. Then

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^n}\, dx = \frac{2\pi}{n\sin(\pi/n)}$$

The length of the real line from -∞ to 0 equals that from 0 to ∞, and $\frac{1}{1+x^n}$ is the same for negative and positive x where n is even. In other words, the function is even, $\int_0^\infty f(x)\, dx = (1/2)\int_{-\infty}^\infty f(x)\, dx$. Therefore, $\int_0^\infty \frac{dx}{1+x^n} = (1/2)\int_{-\infty}^\infty \frac{dx}{1+x^n}$ which we have shown to be equal to $\frac{2\pi}{n\sin(\pi/n)}$, and we can write

$$\int_0^\infty \frac{dx}{1+x^n} = (1/2)\int_{-\infty}^{+\infty} \frac{1}{1+x^n}\, dx = (1/2)\oint_{L+J} \frac{dz}{1+z^n} = \frac{\pi}{n\sin(\pi/n)}$$

(iv) Although the above derivation breaks down when n is odd [why?], use a computer to verify that (15) is nevertheless still true.

If n is odd, then according to Vasco, the roots always contain -1. The reason did not occur to me immediately, but obviously, if p^n = -1, then $(-1)^n$ = -1 for all odd n. Vasco then invokes the General Residue Theorem (p. 419) which applies only to singularities internal to the loop. I don’t find this stated explicitly in the text, but he pointed out in personal communication that an integral passing through a singularity would be unbounded. The loop J+L considered in this problem includes a segment on the real line, so the pole -1 is not internal to the loop and cannot be considered in this derivation, which breaks down for odd n.

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$$\int_0^\infty \frac{1}{1+x^n}\, dx$$

$$\text{ConditionalExpression}\left[\frac{\pi \csc\left(\frac{\pi}{n}\right)}{n}, \left(\text{Re}\left((-1)^{\frac{1}{n}}\right) \leq 0 \bigvee (-1)^{\frac{1}{n}} \notin \mathbb{R}\right) \bigwedge \text{Re}(n) > 1\right]$$

The result in *Mathematica* is $\frac{\pi \csc\left(\frac{\pi}{n}\right)}{n}$ under certain conditions. Since csc(π/n) = 1/sin(π/n), the result is equivalent to the RHS of (15). The conditions appear to be that (Re(p) ≤ 0 or p not real) where Re(n) > 1. In other words, all roots where n > 1 are either complex or they are negative real numbers. More simply, a root of F_n can not be a positive real number.

Letting n = 3, 5, and 7, we see that the equation is true for at least 3 and 7, and perhaps also for 5, but I am not familiar with the meaning of Γ.

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Table[∫_0^∞ 1/(1 + x^n) ⅆx == ((π Csc[π / n]) / n), {n, {3, 5, 7}}]
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$$\left\{\text{True}, \frac{1}{5}\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right) = \frac{\pi}{5\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}}, \text{True}\right\}$$