Needham, Visual Complex Analysis, Chapter 9, Exercise 7, p. 447
G. Palmer, November 23, 2017, 9:40 am PT

Out[35]=

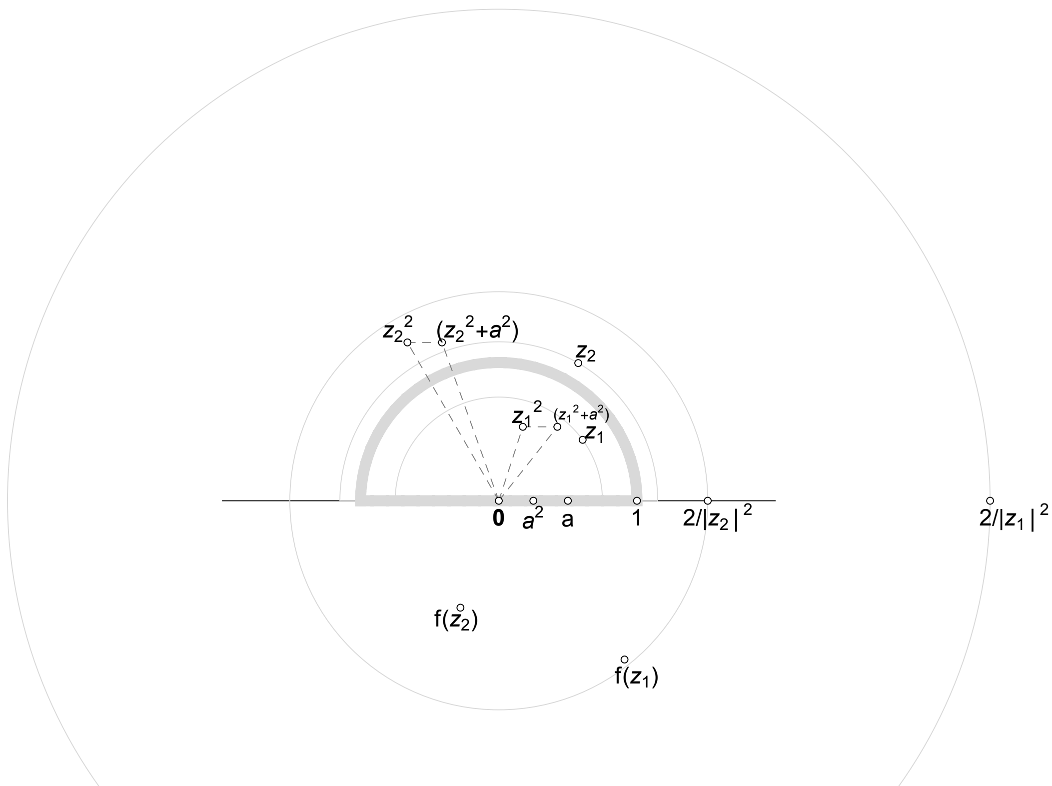

Figure 1. Show $|f(z)| = |1/(z^2+a^2)| < 2/|z|^2$
for sufficiently large |z|

7 Use the result of the previous exercise to do the following problems, in which we assume that a > 0.

(i) Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx = \frac{\pi}{a} e^{-a}$$

Figure 1 is intended to illustrate two situations: $z^2 + a^2$ where z^2 is in the first or the second quadrant. In the first quadrant, we can see that $|z_1^2 + a^2| > |z_1|^2$. In the second quadrant, $|z_1^2 + a^2| < |z_1|^2$. These relations will affect the ratio const./$|z_1|^2$, but the effects are subtle and will be discussed further in showing that |f(z)| < const./$|z_1|^2$.

From Ex. 6, if |f(x)| < $|\text{const.}| / |z|^2$ for sufficiently large |z|, then

$$\int_{-\infty}^{\infty} f(x)\cos x\, dx + i\int_{-\infty}^{\infty} f(x)\sin x\, dx = 2\pi i \sum_{\text{UHP}} \text{Res}[f(z)\, e^{iz}] \qquad (1)$$

Let $f(z) = \frac{1}{x^2+a^2}$ and rewrite it as $\frac{1}{z^2+a^2}$. It is given that a > 0, so a must be a positive real number. Then we wish to establish that $|f(z)| = |\frac{1}{z^2+a^2}| < \text{const.} / |z|^2$.

$$|\frac{z^2+a^2}{z^2+a^2}| < \frac{\text{const.}\,|z^2+a^2|}{|z|^2} \Longrightarrow 1 < \text{const.} \times \left|\frac{z^2+a^2}{z^2}\right| = \text{const.} \times |1+\frac{a^2}{z^2}|.$$

Then the original relation $|f(z)| = |\frac{1}{z^2+a^2}| < \text{const.} / |z|^2$ can be written:

$$\text{const.} > \frac{1}{\left|1+\frac{a^2}{z^2}\right|}$$

This relation must hold if equation (1) is to apply to f(z). We can see that if |z| is sufficiently large, a^2 / z^2 vanishes in the limit and the RHS approaches 1 from above or below. Then, given sufficiently large z, a constant of 2 must be large enough to satisfy the condition for any **a**. While Figure 1 illustrates two such cases. It is not intended as proof. Write f(z), showing that it has two poles, ±ai.

$$f(z) = \frac{1}{(z-ai)(z+ai)}$$

Since ai is the only pole inside the contour L+J, and since ai is a first-order pole, use (11), p. 435, to find the Res[f(z) e^{iz}, ai]. Let P = e^{iz} and let Q = $z^2 + a^2$. Then

$$2\pi i\, \text{Res}[f(z)\, e^{iz}, ai] = 2\pi i \frac{P}{Q'} = 2\pi i \frac{e^{iz}}{2z}\Big|_{ai}$$

$$= 2\pi i \frac{e^{-a}}{2\,ai} = \frac{\pi e^{-a}}{a}$$

By the results in Ex. 6, write

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}\, dx = \frac{\pi e^{-a}}{a}$$

Take just the real parts.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx = \frac{\pi e^{-a}}{a}$$

(ii) Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{\left(x^2+a^2\right)^2} \, dx$$

Let f(x) = $\frac{x}{\left(x^2+a^2\right)^2}$ and rewrite it as f(z) = $\frac{z}{\left(z^2+a^2\right)^2}$. It is given that a > 0, so a must be a positive real number. Then we wish to establish that |f(z)| = $\left|\frac{z}{\left(z^2+a^2\right)^2}\right|$ < const. / | z |^2. Multiply both sides by $\left|\frac{\left(z^2+a^2\right)^2}{z}\right|$.

$$\left|\frac{\left(z^2+a^2\right)^2}{\left(z^2+a^2\right)^2}\right| < \frac{\text{const.}\left|\left(z^2+a^2\right)^2\right|}{|z|\,|z|^2}$$

$$\Longrightarrow 1 < \text{const.} \times \left|\frac{\left(z^2+a^2\right)^2}{z^3}\right| = \text{const.} \times \left|\frac{z^4+2\,a^2\,z^2+a^4}{z^3}\right|$$

$$= \text{const.} \times \left|z + \frac{2\,a^2}{z} + \frac{a^4}{z^3}\right|$$

$$\Longrightarrow \text{const.} > \frac{1}{\left|z+\frac{2\,a^2}{z}+\frac{a^4}{z^3}\right|}$$

Given sufficiently large |z|, this goes to const. > 1/ |z|. Then for any constant greater than max |z|, the relation |f(z)| < const. / | z |^2 will hold and equation (6) can be applied. In practice, this seems to mean that the equation (6) can be applied to any f(z) with reasonably large |z| relative to a. We can see that const. / | z |^2 > (1 / | z |)(1 / | z |^2) = 1 / | z |^3, so |f(z)| < const. / | z |^2 seems safe.

This f(z) has the same two poles as in (i), but they are of order two, so equation (10), p. 435 provides a way to evaluate Res[$\frac{z}{\left(z^2+a^2\right)^2} e^{iz}$, ai] of f(z) on L+J. Rewrite f(z) to display the poles and calculate 2πi Res[f(z) e^{iz}, ai] using (10).

$$\frac{z}{\left(z^2+a^2\right)^2} = \frac{z}{(z-\text{ai})^2\,(z+\text{ai})^2}$$

$$2\pi\text{i Res}\left[\frac{z}{\left(z^2+a^2\right)^2} e^{\text{iz}}, \text{ai}\right] = 2\pi\text{i}\frac{d}{dz}\left[(z-\text{ai})^2 \frac{z}{(z-\text{ai})^2\,(z+\text{ai})^2} e^{\text{iz}}\right]\Big|_{\text{ai}}$$

$$= 2\pi\text{i}\,\frac{d}{dz}\left[\frac{z}{(z+\text{ai})^2} e^{\text{iz}}\right]\Big|_{\text{ai}}$$

$$= 2\pi\text{i}\left[\frac{d}{dz}\left[\frac{z}{(z+\text{ai})^2}\right]e^{\text{iz}} + \frac{z}{(z+\text{ai})^2}\left(\text{i}e^{\text{iz}}\right)\right]\Big|_{\text{ai}}$$

$$=2\pi\text{i}\left[\left(\frac{1}{(z+\text{ai})^2} - 2\,z\,\frac{1}{(z+\text{ai})^3}\right) + \frac{z\text{i}}{(z+\text{ai})^2}\right] e^{\text{iz}}\Big|_{\text{ai}}$$

$$= 2\pi\mathrm{i}\ \left[\left(\frac{1}{(2\ \mathrm{ai})^2} - 2\ \mathrm{ai}\ \frac{1}{(2\ \mathrm{ai})^3}\right) + \frac{-a}{(2\ \mathrm{ai})^2}\right] e^{-a}$$

$$= 2\pi\mathrm{i}\ \left[\left(\frac{1}{(2\ \mathrm{ai})^2} - \frac{1}{(2\ \mathrm{ai})^2}\right) + \frac{-a}{(2\ \mathrm{ai})^2}\right] e^{-a}$$

$$= 2\pi\mathrm{i}\ \frac{1}{4\,a}\ e^{-a} = \frac{\pi\,i}{2\,a}\ e^{-a}$$

Now we can write, using Exercise 6,

$$\int_{-\infty}^{\infty} \frac{x \cos x}{\left(x^2+a^2\right)^2}\, dx + \mathrm{i} \int_{-\infty}^{\infty} \frac{x \sin x}{\left(x^2+a^2\right)^2}\, dx = \frac{\pi\,i}{2\,a}\ e^{-a}$$

Take just the imaginary parts.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{\left(x^2+a^2\right)^2}\, dx = \frac{\pi}{2\,a}\ e^{-a}$$