Needham, Visual Complex Analysis, Chapter 9, Exercise 6, p. 447
G. Palmer, November 7, 2017, 7:45 pm PT

6 Let f(z) be an analytic function with no poles on the real axis, and such that |f(z)| < (const.) / $|z|^2$ for sufficiently large |z|. By integrating f(z) e^{iz} along the contour (L + J) shown in [4a], deduce that

$$\int_{-\infty}^{\infty} f(x) \cos x \, dx + i \int_{-\infty}^{\infty} f(x) \sin x \, dx = 2\pi i \sum_{\text{UHP}} \text{Res}[f(z)\, e^{iz}]$$

[Hint: First show that if y > 0, then $|e^{iz}| < 1$.]

$$e^{iz} = e^{i(x+iy)} = e^{ix}\, e^{-y}$$

$$\Longrightarrow |e^{iz}| = e^{-y} \text{ and if } y > 0, \text{ then } |e^{iz}| < 1.$$

$$2\pi i \sum_{\text{UHP}} \text{Res}[f(z)\, e^{iz}] = \oint_{J+L} f(z)\, e^{iz} dz$$

$$= \oint_J f(z)\, e^{iz} dz + \oint_L f(z)\, e^{iz} dz$$

Since L is effectively the x axis, z traverses the x axis, so allowing |z| = |x| to expand, we can rewrite as follows.

$$\oint_L f(z)\, e^{iz} dz = \int_{-\infty}^{\infty} f(x)\, e^{ix}\, dx = \int_{-\infty}^{\infty} f(x) (\cos x + i \sin x)\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \cos x \; dx + i \int_{-\infty}^{\infty} f(x) \; \sin x \, dx$$

Since $\oint_J f(z)\, e^{iz} dz$ is a contour integral, we can write

$$\left| \oint_J f(z)\, e^{iz} dz \right| \leq |f(z)|\, |e^{iz}|\, \pi\, |z| \quad \text{(because J is a half circle)}$$

We know that $|f(z)| < \frac{\text{const.}}{|z|^2}$ and $|e^{iz}| = e^{-y} < 1$ for y > 0, so

$$\left| \oint_J f(z)\, e^{iz} dz \right| \leq \frac{\text{const.}}{|z|^2}\, \pi\, |z| = \frac{\text{const.}}{|z|}\, \pi$$

Then for sufficiently large |z|, $\oint_J f(z)\, e^{iz} dz \longrightarrow 0$, so we have established that

$$\int_{-\infty}^{\infty} f(x) \cos x \; dx + i \int_{-\infty}^{\infty} f(x) \; \sin x \, dx = 2\pi i \sum_{\text{UHP}} \text{Res}[f(z)\, e^{iz}]$$

for sufficiently large |z|.