Needham, Visual Complex Analysis, Chapter 9, Exercise 5, p. 447
G. Palmer, November 7, 2017, 8:15 am PT

5 If C denotes the unit circle, show that

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta = -\frac{i}{4}\oint_C \frac{\left(z^2-1\right)^2}{z^2(z-2)(2z-1)} dz = \frac{\pi}{4}$$

In previous problems, we have converted contour integrals to real integrals. Here, we can convert a real integral to a contour integral.

Let z = $e^{i\theta}$. Then dz = $ie^{i\theta}d\theta$ and dθ = dz$\big/$ $ie^{i\theta}$.

Substitute the exponential forms for the trig functions.

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta = \int_0^{2\pi} \frac{\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2}{5-4\,\frac{e^{i\theta}+e^{-i\theta}}{2}}\, d\theta$$

Convert to contour integral by substituting z for $e^{i\theta}$ and dz$\big/$ $ie^{i\theta}$ for dθ.

$$= \oint_C \frac{\left(\frac{z-\frac{1}{z}}{2i}\right)^2}{5-4\,\frac{z+\frac{1}{z}}{2}} dz/iz = \oint_C \frac{\frac{\left(z-\frac{1}{z}\right)^2}{-4}}{i\left[5z-4\,\frac{z^2+1}{2}\right]} dz$$

$$= -\frac{i}{4}\oint_C \frac{z^2}{z^2}\,\frac{\left(z-\frac{1}{z}\right)^2}{\left[2z^2-5z+2\right]} dz = -\frac{i}{4}\oint_C \frac{\left(z^2-1\right)^2}{z^2\,(z-2)\,(2z-1)} dz$$

This proves the first equation. Now show that it equals $\frac{\pi}{4}$.

As Vasco has shown, a straightforward approach to the problem is to use (10), p. 345 for the residue with the 0 pole. Then (11) can be used for the residue with the pole 1/2.

$$\text{Res}[f(z), a] = \frac{1}{(m-1)!}\left(\frac{d}{dz}\right)^{m-1}(z-a)^m f(z)\;\Big|_{z=a}$$

Here, m = 2 because z^2 in the denominator of f(z) shows that 0 is a pole of order 2.

$$\text{Res}\left[\frac{\left(z^2-1\right)^2}{z^2\,(z-2)\,(2z-1)}, 0\right] = \frac{1}{1!}\,\frac{d}{dz}\left[z^2\,\frac{(z^2-1)^2}{z^2(z-2)(2z-1)}\right]\Big|_{z=0} = \frac{d}{dz}\left[\frac{\left(z^2-1\right)^2}{(z-2)\,(2z-1)}\right]\Big|_{z=0}$$

Notice that z^2 cancels in the numerator and the denominator. If one omits to cancel, substituting z = 0 produces an undefined result.

$$= \frac{1}{1!}\ \frac{4\,z\left(z^2-1\right)(z-2)\,(2\,z-1) - \left(z^2-1\right)^2[2\,(z-2)+(2\,z-1)\,]}{((z-2)\,(2\,z-1))^2} \qquad \text{by Leibnitz's quotient rule}$$

$$= \frac{0-[2\,(-2)-1]}{(-2)^2\,(-1)^2} = \frac{5}{4}$$

For the pole 1/2, let P = $\left(z^2 - 1\right)^2$ and Q = $z^2\ (z - 2)\ (2\ z - 1)$

$$\text{Res}[\frac{\left(z^2-1\right)^2}{z^2\ (z-2)\ (2\ z-1)}, \frac{1}{2}] = \frac{P}{Q'}\Big|_{\frac{1}{2}} = \frac{\left(z^2-1\right)^2}{\frac{d}{dz}\left[z^2\ (z-2)\ (2\ z-1)\right]}\Big|_{\frac{1}{2}}$$

$$= \frac{\left(z^2-1\right)^2}{2\,(z-2)\,z^2+(2\,z-1)\,z^2+2\,(z-2)\,(2\,z-1)\,z} = \frac{\left(-\frac{3}{4}\right)^2}{2\left(-\frac{3}{2}\right)\left(\frac{1}{4}\right)+0+0} = \frac{\frac{9}{16}}{-\frac{3}{4}} = -\frac{3}{4} \qquad (1)$$

Then we can use (9). Since C makes one wind each around 0 and 1/2, we can write

$$-\frac{i}{4}\int_C f(z)\,dz = -\frac{i}{4}\ 2\pi \text{i}\ (\text{Res}[\text{f}, 0] + \text{Res}[\text{f}, 1/2])$$

$$= \frac{\pi}{2}\left(\frac{5}{4}-\frac{3}{4}\right) = \frac{\pi}{4}$$

What would have happened had we applied (11) to finding the residue on the pole 0? One can see by inspection of the LHS of (1) that the denominator goes to zero at z = 0 unless one can find a z to cancel in the numerator, but the numerator is not rationally divisible by z.