Needham, Visual Complex Analysis, Chapter 9, Exercise 4, p. 446-447
G. Palmer, November 6, 2017, 9:40 am PST

4 The Legendre polynomials $P_n(z)$ are defined by

$$P_n(z) = \frac{1}{2^n\, n!} \frac{d^n}{dz^n} [(z^2 - 1)^n]$$

These polynomials are important in many physical problems, including the quantum mechanical description of the hydrogen atom.

(i) Calculate $P_1(z)$ and $P_2(z)$ and explain why P_n has degree n.

$$P_n(z) = \frac{1}{2^n\, n!} \frac{d^n}{\mathrm{dz}^n} [(z^2 - 1)^n]$$

$$P_1(z) = \frac{1}{2^1\, 1!} \frac{d}{\mathrm{dz}} [(z^2 - 1)^1] = \frac{1}{2} 2\, z\, n\, (z^2 - 1)^{n-1} = \mathrm{z}\, (z^2 - 1)^0 = \mathrm{z}$$

$$P_2(z) = \frac{1}{2^2\, 2!}\; \frac{d}{\mathrm{dz}} [\frac{d}{\mathrm{dz}} [(z^2 - 1)^2]]$$

$$= \frac{1}{8} \frac{d}{\mathrm{dz}} [4\, z\, (z^2 - 1)] = \frac{1}{8} (\, 8\, z^2 + 4\, (z^2 - 1))$$

$$= z^2 + \frac{z^2}{2} - \frac{1}{2} = \frac{1}{2}\, (3\, z^2 - 1)$$

Evidently, P_n has degree n because because n is the highest power of the terms. One could say it has degree n because it undergoes n derivations each of which contributes a factor of z owing to the z^2 factor in parentheses.

(ii) Use (6) to show that

$$P_n(z) = \frac{1}{2\pi i} \oint_K \frac{(Z^2 - 1)^n}{2^n\, (Z - z)^{n+1}}\, dZ$$

(6) gives an expression for the nth derivative of f at some point a.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad (6)$$

In (i), above, we are given $P_n(\mathrm{z})$ in terms of a constant times the nth derivative of $(z^2 - 1)^n$:

$P_n(z) = \frac{1}{2^n\, n!} \frac{d^n}{\mathrm{dz}^n} [(z^2 - 1)^n]$, which could be rewritten as $\frac{1}{n!} \frac{d^n}{\mathrm{dz}^n} [\frac{1}{2^n}\, (z^2 - 1)^n] = \frac{1}{n!} f^{(n)}(z)$

i.e. $\frac{1}{n!}$ times the nth derivative of $\frac{1}{2^n}\left(z^2-1\right)^n$ or $\frac{1}{n!}f^{(n)}$(z). By substituting Z for z, we obtain f(Z) = $\frac{1}{2^n}\left(Z^2-1\right)^n$, which can be substituted for f(z) in (6). Then, substituting K for L and (Z-z) for (z-a) in (6), we obtain

$$\frac{1}{n!}f^{(n)}(z) = \frac{1}{n!}\frac{n!}{2\pi i}\oint_K \frac{\left(Z^2-1\right)^n}{2^n (Z-z)^{n+1}}\,dZ = \frac{1}{2\pi i}\oint_K \frac{\left(Z^2-1\right)^n}{2^n (Z-z)^{n+1}}\,dZ = P_n(z)$$

What does it mean? The Z notation appears to refer to values on a loop that encloses a point z (pp. 432-433, 442-443). The illustrations [3] and [6] show Z on a circle C, as in [3], or on a loop K, as in [6]. Both loops enclose the anchor point of z. f(Z) appears to be analytic. Our f(Z) appears to have 2n zeroes and n+1 singularities. The general structure is just the nth order derivative of Cauchy's Formula (p. 427) at a point z. The nth derivative of $\frac{1}{n!}$f(Z) = $\frac{1}{n!\,2^n}\left(Z^2-1\right)^n$ is the Legendre Polynomial. What is the significance of $\left(Z^2-1\right)^n$?

(iii) By taking K to be a circle of radius $\sqrt{\left|z^2-1\right|}$ centred at z, deduce that

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \left(z + \sqrt{\left|z^2-1\right|}\,\cos\theta\right)^n d\theta$$

Here we follow Vasco's suggestion and let $\rho = \sqrt{z^2-1}$, rather than $\sqrt{\left|z^2-1\right|}$.

Let $\rho = \sqrt{z^2-1}$

Then $\rho^2 = z^2 - 1$

$Z = \rho\, e^{i\theta} + z$,

$dZ = i\rho e^{i\theta}\, d\theta$,

$f(Z) = \left(Z^2-1\right)^n = (\rho^2 e^{i2\theta} + 2\,\rho e^{i\theta} z + z^2 - 1)^n$

$(Z - z) = \rho\, e^{i\theta}$

We can now use the result from (ii) $P_n(z) = \frac{1}{2\pi i}\oint_K \frac{\left(Z^2-1\right)^n}{2^n (Z-z)^{n+1}}\,dZ$ and parameterize it with θ and our value of Z. The range of integration is written from 0 to π rather than 0 to 2π, so the result must be doubled to obtain the sum for a full rotation. We can do this because the expression z + $\sqrt{z^2-1}$ cos θ is symmetric about $\pi = \theta$ (Vasco, Forum, p.c.).

$$P_n(z) = \frac{2}{2\pi i}\int_0^{\pi} \frac{\left(\rho^2 e^{i2\theta} + 2\rho e^{i\theta} z + z^2 - 1\right)^n}{2^n \left(\rho e^{i\theta}\right)^{n+1}} \, i\rho e^{i\theta}\, d\theta$$

We see that $\frac{dZ}{\rho} = id\theta$, so we cancel the "i"s in the numerator and denominator and subtract 1 from the exponent of $\rho\, e^{i\theta}$. Now numerator and denominator have the same power and we can separate terms.

$$P_n(z) = \frac{1}{\pi} \oint_K \left(\frac{\rho^2\, e^{i2\theta} + 2\,\rho e^{i\theta}\, z + z^2 - 1}{2\,\rho\, e^{i\theta}}\right)^n d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(\frac{\rho\, e^{i\theta}}{2} + z + \frac{z^2\, e^{-i\theta}}{2\,\rho} - \frac{e^{-i\theta}}{2\,\rho}\right)^n d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(z + \frac{\rho\, e^{i\theta}}{2} + \frac{\left(z^2 - 1\right)\, e^{-i\theta}}{2\,\rho}\right)^n d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(z + \frac{\rho\, e^{i\theta}}{2} + \frac{\rho\, e^{-i\theta}}{2}\right)^n d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(z + \frac{\rho\,\left(e^{i\theta} + e^{-i\theta}\right)}{2}\right)^n d\theta = \frac{1}{\pi}\int_0^{\pi} \left(z + \sqrt{z^2 - 1}\, \cos\theta\right)^n d\theta$$

(iv) Check that this last formula yields the same $P_1(z)$ and $P_2(z)$ as you obtained in part (i).

$$P_1(z) = \frac{1}{\pi}\int_0^{\pi} \left(z + \sqrt{z^2 - 1}\, \cos\theta\right)^1 d\theta$$

$$= \frac{1}{\pi}\left[z\,\theta + \sqrt{z^2 - 1}\, \sin\theta\right]_0^{\pi} = \frac{1}{\pi}[\,\pi\, z\,] = z$$

$$P_2(z) = \frac{1}{\pi}\int_0^{\pi} \left(z + \sqrt{z^2 - 1}\, \cos\theta\right)^2 d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(z^2 + 2\, z\sqrt{z^2 - 1}\, \cos\theta + \left(z^2 - 1\right)\, \cos^2\theta\right)^2 d\theta$$

$$= \frac{1}{\pi}\left[z^2\,\theta + 2\, z\sqrt{z^2 - 1}\, \sin\theta + \left(z^2 - 1\right)\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\pi\, z^2 + 0 + \frac{\pi}{2}\left(z^2 - 1\right)\right) - (0 + 0 + 0)\right]$$

$$= z^2 + \frac{1}{2}\left(z^2 - 1\right) = \frac{3}{2}\, z^2 - \frac{1}{2} = \frac{1}{2}\left(3\, z^2 - 1\right)$$

The values $P_1(z) = \mathrm{z}$ and $P_2(z) = \frac{1}{2}(3\ z^2\text{-}1)$ are the same as in part (i).