Needham, *Visual Complex Analysis*, Chapter 9, Exercise 3, p. 446
G. Palmer, October 24, 2017, 9:20 am PDT

3 (i) Show that if C is any simple loop round the origin, then

$$\binom{n}{r} = \frac{1}{2\pi i}\oint_C \frac{(1+z)^n}{z^{r+1}}dz$$

By writing the numerator as a Taylor series, we get terms that are more simple than $(1+z)^n$ and can serve as the basis for a Laurent series.

$$(1+z)^n = 1 + \text{nz} + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + ... \qquad \text{(from (14), p. 73)}$$

$$\text{f(z)} = \frac{(1+z)^n}{z^{r+1}} = \frac{1}{z^{r+1}} + \frac{\text{nz}}{z^{r+1}} + \frac{n(n-1)}{2!\,z^{r+1}}z^2 + \frac{n(n-1)(n-2)}{3!\,z^{r+1}}z^3 + ...$$

$$= \frac{1}{z^{r+1}} + \frac{n}{z^r} + \frac{n(n-1)}{2!\,z^{r-1}} + \frac{n(n-1)(n-2)}{3!\,z^{r-2}} + ... + \frac{n(n-1)(n-2)...(n-(r-1))}{r!\,z} + ...$$

[Power of (1/z) = 1 after r subtractions from (r+1), so coefficient in denominator reaches r! and last factor in numerator of (1/z) term subtracts (r-1).]

We need only the residue of the 1/z term of the series.

$$\text{Res[f(z), 0]} = \frac{n(n-1)(n-2)...(n-(r-1))}{r!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r} \qquad (1)$$

by multiplying top and bottom by (n-r)! and Exercise 32, Ch 1, p. 50.

Then, from III Calculus of Residues, 1 Laurent series centred at a pole, p. 435, top,

$$\oint_C \text{f(z) dz} = 2\,\pi\,\text{i Res[f, a]}$$

can be adapted here as

$$\oint_L \text{f(z) dz} = 2\,\pi\,\text{i Res[f, 0]}$$

because f(z) has a simple pole at 0. Then from (1),

$$\frac{1}{2\pi i}\oint_C \frac{(1+z)^n}{z^{r+1}}\text{dz} = \frac{1}{2\pi i}\,2\,\pi\,i\,\text{Res[f(z), 0]} = \binom{n}{r}$$

(ii) By taking C to be the unit circle, deduce that

$$\binom{2n}{n} \leq 4^n.$$

From (i), $\binom{n}{r} = \frac{1}{2\pi i} \oint_C \frac{(1+z)^n}{z^{r+1}} dz$, so $\binom{2n}{n} = \frac{1}{2\pi i} \oint_C \frac{(1+z)^{2n}}{z^{n+1}} dz$.

By (5), p. 387, we know that $\left| \int_K f(z)\, dz \right| \leq$ M · (length of K). *Let* $f(z) \equiv \frac{(1+z)^{2n}}{z^{n+1}}$.

Since z lies on the unit circle and makes one traversal of C, by (5), p. 387, and since $\binom{2n}{n}$ is always positive:

$$\binom{2n}{n} = \left| \frac{1}{2\pi i} \oint_C \frac{(1+z)^{2n}}{z^{n+1}} dz \right| = \left| \frac{1}{2\pi i} \right| \left| \oint_C (1+z)^{2n} dz \right| \leq \frac{1}{2\pi}\, 2^{2n}\, 2\pi \cdot 1 = 4^n$$