Needham, *Visual Complex Analysis*, Chapter 9, Exercise 2
G. Palmer, October 7, 2017, 9 am PDT

2 Let f(z) be analytic on and inside a circle K defined by |z-a| = ρ, and let M be the maximum of |f(z)| on K.

(i) Use (6) to show that $\left| f^{(n)}(a) \right| \leq \frac{n!\,M}{\rho^n}$.

Given $f^{(n)}(a) = \frac{n!}{(2\pi i)} \oint_L \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{dz}$ (6) in Needham

We can substitute our parameters into (6) as

$$f^{(n)}(a) = \frac{n!}{(2\pi i)} \oint_K \frac{f(z)}{\rho^{n+1}}\,\mathrm{dz} = \frac{n!}{(2\pi i)\,\rho^{n+1}} \oint_K f(z)\,\mathrm{dz}. \qquad (1)$$

$\left|\oint_K f(z)\,\mathrm{dz}\right| \leq$ len(K) M, where M = Max$|f(z)|$ from p. 433 $|\int$f(z)dz$|$ ≤ len(path) Max(|f(z)|)

then

$$|f^{(n)}(a)| \leq \left| \frac{n!}{(2\pi i)} \right| 2\pi\rho\,\mathrm{M} = n!\rho\,\frac{M}{\rho^{n+1}} = n!\,\frac{M}{\rho^n}$$

(ii) Suppose |f(z)| ≤ M for all z, where M is some constant. By putting n = 1 in the above inequality, rederive Liouville's Theorem [p. 360].

Liouville's Theorem: An analytic mapping cannot compress the entire plane into a region lying inside a disc of finite radius without crushing it all the way down to a point.

Answer:

|z-a| = ρ is a circle of finite radius centred at a.

Put n = 1 in the inequality:

$$|f^{(1)}(a)| = |f'(a)| \leq \frac{M}{\rho}$$

Then we note that this has the same coefficient $\frac{M}{\rho}$ as the Liouville equation for f(p) that leaves the origin fixed, i. e. |f(p)| = $\frac{M}{N}$|p|, where the meaning of M is the same as in this problem, and the meaning of N as the preimage radius is the same as ρ in this problem. We also note that the problem statement stipulates that M is a constant. That point was unclear in the discussion of the Liouville Theorem on p. 360, but it is necessary. At this point Vasco has: *As ρ increases this result remains true and in the limit as ρ goes to infinity*[;] *|f '(a)| = 0 for all a, and so we can replace a by z and write f(z) is a constant. Done*. Since f(z) is everywhere a constant, the entire disc is crushed to a single point.

(iii) Suppose that |f(z)| ≤ M $|z|^n$ for all z, where n is some positive integer. Show that $f^{(n+1)}(z) \equiv 0$, and deduce that f(z) must be a polynomial whose degree does not exceed n.

Using equation (1) in part (i), we can write

$$f^{(n+1)}(a) = \frac{(n+1)!}{(2\pi i)\,\rho^{n+2}} \oint_K f(z)\,\mathrm{dz}$$

$$|f^{(n+1)}(a)| \leq \left|\frac{(n+1)!}{(2\pi i)\,\rho^{n+2}}\right| 2\pi\rho \mathrm{M} = \frac{(n+1)!\,M}{\rho^{n+1}}$$

Since M is a constant, the inequality holds no matter how large we make ρ. Since the RHS goes to zero as ρ goes to infinity, $f^{(n+1)}$(a) = 0 for all a. Following Vasco, we can write this $f^{(n+1)}$(a) ≡ 0 and replace a with z to write $f^{(n+1)}$(z) ≡ 0.

This indicates that $f^{(n)}$(z) = c, a constant., or a polynomial of degree 0. Integrating n more times results in a function $f^{(n-n)} = f^{(0)}$ = f, which, assuming the coefficient of the term containing z^n is non-zero, is a polynomial of degree n, meaning the term of highest degree has degree n. Then a polynomial P^{n+1} would have the (n+1) derivative $\left(P^{n+1}\right)^{(n+1)}$ = P^0 = const. rather than 0. Hence, f(z) must be a polynomial whose degree does not exceed n.