Needham, Visual Complex Analysis, Chapter 9, Exercise 13, p. 449
G. Palmer, Jan 5, 2017, 1:45 pm PT

13 (i) Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^2-n^2} = \frac{\pi \cot \pi z}{z}$$

If we could apply (13) directly to this problem, we would write

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^2-n^2} = -\pi \sum_{\substack{\text{poles} \\ \text{of } f(z)}} \text{Res}[\frac{1}{z^2-n^2}\cot(\pi z)]$$

but this encounters the difficulty that "if any of the poles of f(z) happen to be integers, then these values are understood to be excluded from the LHS of (13)" (p. 441). In this f(z), the poles ±n are integers and all the integers are poles! Another approach is needed.

We assume that the problem is based in diagram [5], p. 440. In order to apply (13), one must establish that $|\frac{1}{z^2-n^2}|$ is an analytic function such that $|\frac{1}{z^2-n^2}| <$ (const.) $/\,|z|^2$. Since we are using the model of [5], |z| > N and $|z|^2 > N^2$ for all z. Vasco points out that $|z^2 - w^2|$ = |z+w| |z-w|, which is the product of the distance from w to -z by the distance from w to z. This product is greater than $|z|^2$ by the Triangle Inequality and the definition of [5], so it follows that $|\frac{1}{z^2-n^2}| <$ (const.) $/\,|z|^2$ given sufficiently large z and z ≠ n. We will show that the criterion remains valid through the rewrite and calculations involved in this answer.

On p. 441, Needham presents an approach using (13) that we may be able to use. He presents the function f(z) = $1 \big/ (z-w)^2$, which contains a double pole w which is an arbitrary non-integer. Then he uses (13) and calculates the residue. Perhaps $\frac{1}{z^2-n^2}$ can be rewritten as a function with an arbitrary non-integer pole. In order to show that the rewrite z has a different meaning from our original z, I'm going to use capital letters for the W and Z in the rewrite format. Since z can be defined as a non-integer complex number simply be avoiding the integers, replace z with W, and on the RHS replace n with the complex number Z.

$$\sum_{n=-\infty}^{\infty} \frac{1}{W^2-n^2} = -\pi \text{Res}\left[\frac{\cot(\pi z)}{W^2-Z^2},\ W\right] \qquad (1)$$

By the same reasoning as above, since $|W^2 - Z^2| > |Z|^2$, the size of |f| criterion should remain valid for (1). Find the residue following (11), by differentiating in terms of Z, as Needham does in the example on p. 441. *Ultimately, this means one differentiates on the variable which has been substituted for the integer n.*

$-\pi\ \Sigma\ \text{Res}[\frac{\cot(\pi z)}{W^2-Z^2}, \{W, -W\}]$, W not an integer

$$= -\pi\ [\frac{P(W)}{Q'(W)} + \frac{P(-W)}{Q'(-W)}] = -\pi\ [\frac{\cot(\pi Z)}{-2Z} + \frac{\cot(\pi Z)}{-2Z}]_{n = W, -W}$$

$$= -\pi\ [\frac{\cot(\pi W)}{-2W} + \frac{\cot(\pi(-W))}{-2(-W)}] = \frac{\pi \cot(\pi W)}{W}, \quad W \neq 0$$

Rewrite the equation by reversing the original substitution. That is, W → z.

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^2-n^2} = \frac{\pi \cot \pi z}{z}$$

This is what we wanted. But something odd has been done here. When we substituted and calculated the residue on W, we abandoned the idea that z could take an integer value. The essence of the method is to find the residue by differentiating on the variable Z, which replaced n, and then substituting the non-integer pole (W, which replaced z) into the result. Since one can not differentiate a continuous variable, Z must be regarded as continuous, but we will only instantiate it with integer values (n).

(ii) Show that the previous equation $[\sum_{n=-\infty}^{\infty} \frac{1}{z^2-n^2} = \frac{\pi \cot \pi z}{z}]$ *can be rewritten as*

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2\pi^2}$$

Rearrange the equation from (i).

$$\cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{z}{\pi(z^2-n^2)} = \sum_{n=-\infty}^{\infty} \frac{\pi z}{\pi^2 z^2-\pi^2 n^2}$$

Substitute z for πz.

$$\cot(z) = \sum_{n=-\infty}^{\infty} \frac{z}{z^2-\pi^2 n^2}$$

Rewrite to sum for n = 1..∞ and multiply the sum by 2. Then add the zero term of the cotangent series, which is $\frac{1}{z}$ (see the last line of p. 438 and substitute z for πz). One could also derive this term from $\frac{z}{z^2-\pi^2 n^2}|_{n=0}$, keeping in mind that it does not need to be multiplied by 2, because there is no counterpart term in -N.

$$\cot(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-\pi^2 n^2}$$

This is what we wanted.

(iii) Show that the previous equation can be rewritten as

$$\frac{d}{dz}[ln(\sin z\,/\,z)] = \sum_{n=1}^{\infty} \frac{d}{dz}\, ln\left(z^2 - n^2\,\pi^2\right)$$

Rewrite (ii)

$$\cot(z) - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2\,z}{z^2 - n^2\,\pi^2}$$

Integrate both sides:

$$\int \cot(z)\,dz - \int \frac{1}{z}\,dz = \sum_{n=1}^{\infty} \int \frac{2\,z}{z^2 - n^2\,\pi^2}\,dz \quad \text{(integral of sum = sum of integrals)}$$

$$\ln(\sin(z)) - \ln(z) = \sum_{n=1}^{\infty} \ln\left(z^2 - n^2\,\pi^2\right)$$

$$\ln(\sin(z)/z) = \sum_{n=1}^{\infty} \ln\left(z^2 - n^2\,\pi^2\right) \qquad \text{(LHS: ln of product is sum of ln's)}$$

Constants of integration need not be written because they disappear under differentiation of both sides.

$$\frac{d}{dz}\ln(\sin(z)/z) = \sum_{n=1}^{\infty} \frac{d}{dz}\ln\left(z^2 - n^2\,\pi^2\right) \quad \text{(deriv. of sum = sum of derivatives)}$$

This is what we wanted.

(iv) By integrating along any path from 0 to z that avoids integers, and then exponentiating both sides of the resulting equation, deduce that

$$\sin z = z\left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\,\pi^2}\right)\left(1 - \frac{z^2}{3^2\,\pi^2}\right)\cdots.$$

Why is a definite integral required for this problem? Perhaps because it is desireable for the RHS to converge to a definite value between 0 and 1. Using an indefinite integral in this case leads to an expression that does not obviously converge, as it multiplies every factor by $\left(-n^2\,\pi^2\right)$, which is equivalent to a single factor of $\left(-n^2\,\pi^2\right)^{\infty}$.

Vasco has pointed out that the problem statement probably contains a mistake and should probably read *By integrating along any path from 0 to z that avoids integer multiples of* π^2. That is, the path should not pass through poles.

Integrate both sides on 0 to Z, where $Z \notin \mathbb{Z}$.

$$\int_0^Z \frac{d}{dz}\ln(\sin(z)/z)\, dz = \int_0^Z \sum_{n=1}^{\infty} \frac{d}{dz} \ln\left(z^2 - n^2\pi^2\right) dz$$

$$\ln(\sin(z)/z)\big|_0^Z = \sum_{n=1}^{\infty} \ln\left(z^2 - n^2\pi^2\right) \big|_0^Z$$

Find the LHS.

$$\ln(\sin(z)/z)\big|_0^Z = \ln(\sin(Z)/Z) - \ln(1) \qquad \text{(by the hint)}$$

$$= \ln(\sin(Z)/Z) \qquad (1)$$

Find the RHS.

$$\sum_{n=1}^{\infty} \ln\left(z^2 - n^2\pi^2\right) \big|_0^Z = \sum_{n=1}^{\infty} \ln\left(Z^2 - n^2\pi^2\right) - \sum_{n=1}^{\infty} \ln\left(-n^2\pi^2\right)$$

$$= \sum_{n=1}^{\infty} \ln\left[\left(Z^2 - n^2\pi^2\right)/(-n^2\pi^2)\right]$$

$$= \sum_{n=1}^{\infty} \ln\left[-n^2\pi^2\left(1 - \frac{Z^2}{n^2\pi^2}\right)/(-n^2\pi^2)\right]$$

$$= \sum_{n=1}^{\infty} \ln\left(1 - \frac{Z^2}{n^2\pi^2}\right) \qquad (2)$$

Rewrite the equation using (1) and (2).

$$\ln(\sin(Z)/Z) = \sum_{n=1}^{\infty} \ln\left(1 - \frac{Z^2}{n^2\pi^2}\right)$$

Replace Z with z.

$$\ln(\sin(z)/z) = \sum_{n=1}^{\infty} \ln\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

$$= \ln \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) \qquad \text{(sum of ln = ln of product)}$$

Exponentiate both sides.

$$\sin(z)/z = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\pi^2}\right)\left(1 - \frac{z^2}{3^2\pi^2}\right)\cdots$$

$$\sin(z) = z\left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{2^2\pi^2}\right)\left(1 - \frac{z^2}{3^2\pi^2}\right)\cdots$$