Needham, Visual Complex Analysis, Chapter 9, Exercise 12, p. 448
G. Palmer, December 7, 2017, 12:15 pm PT

Use the result of the previous question to do the following:

(i) Show that $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \frac{\pi^2}{12}$

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \sum_{n=-\infty}^{\infty} (-1)^n \left(-\frac{1}{n^2}\right) - \left[1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots\right]$$

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \sum_{n=-\infty}^{\infty} (-1)^n \left(-\frac{1}{n^2}\right) - \sum_{n=-\infty}^{-1} (-1)^n \left(-\frac{1}{n^2}\right)$$

Notice that we have excluded the pole 0 of $-1/n^2$ as required in Ex. 11. We can see that the sum from -∞ to -1 equals the sum from 1 to ∞, so we can write the RHS as

$$\sum_{n=1}^{\infty} (-1)^n \left(-\frac{1}{n^2}\right) = \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \left(-\frac{1}{n^2}\right) \qquad (1)$$

We write the function as $f(z) = \left(-\frac{1}{z^2}\right)$.

$$\sum_{n=-\infty}^{\infty} (-1)^n \left(-\frac{1}{n^2}\right) = -\pi \underset{\substack{\text{poles} \\ \text{of } f(z)}}{\operatorname{Res}} [f(z) \csc(\pi z)] \qquad (2)$$

We saw in (11) that $\csc(\pi(n+z)) = (-1)^n \csc(\pi z)$.

From Ex. (11), $\csc(z) = \frac{1}{\pi z} + \frac{(\pi z)^2}{6} + \frac{7(\pi z)^4}{360} + \cdots$

$$g(z) = \left(-\frac{1}{z^2}\right) \csc(\pi z) = -\frac{1}{\pi z^3} - \frac{(\pi z)^2}{6(\pi z^3)} - \frac{7(\pi z)^4}{360(\pi z^3)} - \cdots$$

$$= -\frac{1}{\pi z^3} - \frac{\pi}{6z} - \frac{7\pi^3 z}{360} - \cdots$$

$$\operatorname{Res}[f(z)\csc(\pi z), 0] = \operatorname{Res}[g(z), 0] = -\frac{\pi}{6}$$

Since f(z) has only the one pole, 0, equation (2) becomes

$$\sum_{n=-\infty}^{\infty} (-1)^n \left(-\frac{1}{n^2}\right) = -\pi \operatorname{Res}[f(z)\csc(\pi z), 0]$$

$$= -\pi \left(-\frac{\pi}{6}\right) = \frac{\pi^2}{6}$$

Then, from (1),

$$\sum_{n=1}^{\infty} (-1)^n \left(-\frac{1}{n^2}\right) = -\frac{\pi}{2} \text{ Res[f(z) csc}(\pi z), 0] = \frac{\pi^2}{12}$$

(ii) Find the sum of the series

$$\frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} + \cdots \frac{(-1)^{n+1}}{(n^2+1)}$$

$$\frac{(-1)^{n+1}}{(n^2+1)} = (-1)^n \frac{-1}{(n^2+1)}$$

Thus, it appears that the cosecant equation from Ex. 11 can be applied.

Let $f(z) = \frac{-1}{(z^2+1)}$

The poles of f(z) are {-i, +i}. Rewrite f(z) as the sum of two partial fractions:

Let $f(z) = \frac{-1}{(z^2+1)} = -(\frac{1}{(z+i)(z+i)}) = -\left(\frac{i}{2(z+i)} - \frac{i}{2(z-i)}\right) = \frac{i}{2(z-i)} - \frac{i}{2(z+i)}$

$$= \frac{i}{2z\left(1-\frac{i}{z}\right)} - \frac{i}{2z\left(1-\left(-\frac{i}{z}\right)\right)}$$

Find a series for the partial sums:

for $|z| > 1$ for $|z| > 1$

$$= \frac{i}{2z}\left[1 + \frac{i}{z} + \left(\frac{i}{z}\right)^2 + \left(\frac{i}{z}\right)^3 + \cdots\right] - \frac{i}{2z}\left[1 - \frac{i}{z} - \frac{1}{z^2} + \frac{i}{z^3} + \cdots\right]$$

$$= \left[\frac{i}{2z} - \frac{1}{2z^2} - \frac{i}{2z^3} + \frac{1}{2z^4} + \cdots\right] - \left[\frac{i}{2z} + \frac{1}{2z^2} - \frac{i}{2z^3} - \frac{1}{2z^4} + \cdots\right]$$

$= -\frac{i}{z^3} + \cdots$? We will hope this is a good approximation

Find a Laurent series for $g(z) = f(z)\csc(\pi z)$:

$$-\frac{i}{z^3}\left[\frac{1}{\pi z} + \frac{(\pi z)^2}{6} + \frac{7(\pi z)^4}{360} + \cdots\right] = -\frac{i}{\pi z^4} - \frac{\pi^2}{6z} - \cdots$$

Then

$$\text{Res}[\text{g(z)}, p]_{\{-i,i\}} = \text{Res}[\text{f(z)csc}(\pi \text{z}), p]_{\{-i, i\}} = \left(-\frac{\pi^2}{6}\right)$$

The series $\sum_{n=-\infty}^{\infty}(-1)^n \frac{-1}{(n^2+1)}$ does not permit excluding an integer pole, because the poles of f(z) are {-i, i}, which are not integers. Therefore, to find the value of $\sum_{n=1}^{\infty}(-1)^n \frac{-1}{(n^2+1)}$, we first subtract the value of the n=0 term from the sum $\sum_{n=-\infty}^{\infty}$. The subtracted term equals (-1). Then the remainder of the infinite series can be divided by two.

$$\sum_{n=1}^{\infty}(-1)^n \frac{-1}{(n^2+1)} = \frac{1}{2}([\sum_{n=-\infty}^{\infty}(-1)^n \frac{-1}{(n^2+1)}] - 1)$$

$$\sum_{n=-\infty}^{\infty}(-1)^n \frac{-1}{(n^2+1)} = 2(\sum_{n=1}^{\infty}(-1)^n \frac{-1}{(n^2+1)}) + 1 \qquad (3)$$

residues of g(z) on n = - residues of g(z) on poles of f(z))
multiply both sides by π

$$\sum_{n=-\infty}^{\infty}(-1)^n \frac{-1}{(n^2+1)} = -\pi\left(-\frac{\pi^2}{6}\right) = \frac{\pi^3}{6} \qquad \text{from Ex. 11}$$

$$2(\sum_{n=1}^{\infty}(-1)^n \frac{-1}{(n^2+1)}) + 1 = \frac{\pi^3}{6} \qquad \text{by substitution from (3)}$$

$$\sum_{n=1}^{\infty}(-1)^n \frac{-1}{(n^2+1)} = \frac{\pi^3}{12} - \frac{1}{2} = 2.08386...$$