Needham, Visual Complex Analysis, Chapter 9, Exercise 11, p. 448
G. Palmer, December 6, 2017, 1:45 pm PT

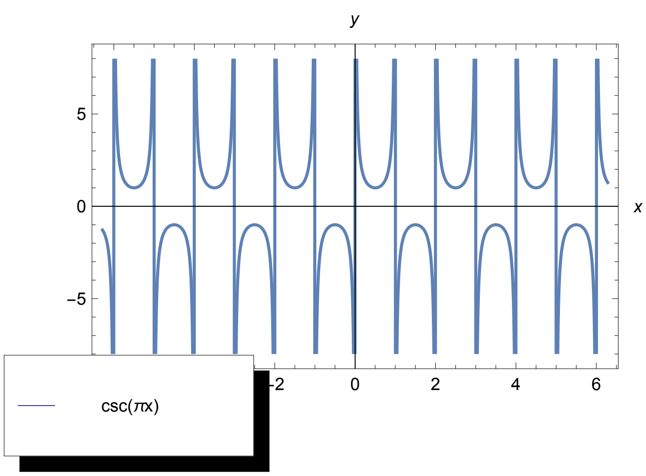
11 Show that if f(z) is an analytic function such that $|f(z)| < (\text{const.}) / |z|^2$ *for sufficiently large* $|z|$*, then*

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{\text{poles of } f(z)} Res[f(z)\ cosec(\pi z)]$$

In this formula, it is understood that if any of the poles of f(z) happen to be integers, then these values of n are excluded from the LHS.

Prolog

I at first thought that the factor $(-1)^n$ would be derived from a series with alternating sign. This turned out not to be the case. The following plot of $\csc(\pi\ x)$ shows that, on the real line, the sign alternates with period n = 2. There are two singularities, one negative and one positive, at every integer and 0, approaching from opposite directions. The function $\csc(\pi\ z)$ is odd (i.e. f(z) = -f(-z)). Cosecant will hereafter be abbreviated csc in equations.



Answer

Find a Laurent series and residue for csc(πz).

$$\csc(\pi z) = \frac{1}{\pi z}\ [1 + \frac{1}{6}\ (\pi z)^2 + \frac{1}{360}\ 7\ (\pi z)^4 + \ldots]$$

(From Vasco's expansion of second bracket in Ex 10)

$$\text{Res}[\csc(\pi z), 0] = \frac{1}{\pi} \qquad (1)$$

The factor $(-1)^n$, which appears in the equation to be proven, seems to be an inherent quality of the cosecant arising from the the fact that $\cos(\pi n)$ can be written $(-1)^n$. Vasco has provided the key to obtaining the factor in his answer. In our model of S, z increases by 1 each time it encircles an additional pole, hence we can write trig functions with arguments of π (n+z). In the following, I have substituted z for Vasco's ξ, which seems justifiable since we are not considering derivatives and infinitesimal values. To my mind, this makes the substitution into equation (3) easier to follow.

$$\sin(\pi(n+z)) = \sin(\pi n)\cos(\pi z) + \cos(\pi n)\sin(\pi z)$$

$$= \cos(\pi n)\sin(\pi z) = (-1)^n \sin(\pi z)$$

$$\csc(\pi(n+z)) = \frac{1}{\sin(\pi z)}(-1)^n = (-1)^n \csc(\pi z) \qquad (2)$$

We will see that $\csc(\pi(n+z))$ arises in Res[g(z), n] in the derivation of equation (3).

We first need to establish that g(z) vanishes with increasing N in order to write equations like those on pp. 440-441 and arrive at the desired equation.

It is given that f(z) is analytic and $|f(z)| < (\text{const.}) / |z|^2$. For n > 1, |z| > N on S, so $1/|z|^2 < 1/N^2$. Then, find $|\csc(\pi z)|$ on the horizontal edges of S. Since $y = \pm\left(N + \frac{1}{2}\right)$, $\lim_{(|z|=N)\to\infty} |\csc(\pi z)| \to 0$, so for sufficiently large N, $|\csc(\pi z)| < 1$, as shown by the following:

$$|\csc(\pi z)| = |1/\sin(\pi z)| = 1/\left|\left(e^{i\pi z} - e^{-i\pi z}\right)\right| = 1/\left|e^{-y}e^{i\pi x} - e^{y}e^{-i\pi x}\right|$$

$$= 1/\left|e^{-N}e^{i\pi x} - e^{N}e^{-i\pi x}\right| \to 1/|-e^N| = 1/\left|e^N\right| < 1$$

On the vertical edges, $z = \pm(N+\frac{1}{2}) + iy$.

$$|\csc(\pi z)| = 1/\left|\left(e^{i\pi\left(\pm\left(N+\frac{1}{2}\right)+iy\right)} - e^{-i\pi z\left(\pm\left(N+\frac{1}{2}\right)+iy\right)}\right)\right|$$

$$= 1/\left|\left(e^{\pm i\pi\left(N+\frac{1}{2}\right)-\pi y} - e^{\mp i\pi\left(N+\frac{1}{2}\right)+\pi y}\right)\right|$$

One can see from [5] that y increases with increasing N. The denominator is dominated by $|-e^{\pi y}|$, so $(1/|-e^{\pi y}|)$ also vanishes as N tends to infinity. For sufficiently large N, we have established that $|\csc(\pi z)|$ vanishes on S.

$$|g(z)dz| \leq (\text{Max } g(z) \text{ on S})\,(\text{perimeter of S}) < \left(1/N^2\right)\cdot 0 \cdot (8N+4)$$

for sufficiently large N. The RHS tends to 0.

Because f(z) is analytic and |f(z)| < (const.) / $|z|^2$, this argument applies to $\oint_S$ f(z) csc(πz) dz. The derivation follows the pattern presented on p. 440 and 441.

$$0 = \lim_{N\to\infty} \frac{1}{2\pi i} \oint_S f(z)\csc(\pi z)\,dz = \sum_{\text{all poles}} \text{Res}[f(z)\csc(\pi z)]$$

$$= \sum_{n=-\infty}^{\infty} \text{Res}[f(z)(-1)^n \csc(\pi z),\ n] + \sum_{\text{poles of } f(z)} \text{Res}[f(z)\csc(\pi z)] \quad [(-1)^n \text{ from } (2)]$$

$$= \frac{1}{\pi}\sum_{n=-\infty}^{\infty} (-1)^n f(n) + \sum_{\text{poles of } f(z)} \text{Res}[f(z)\csc(\pi z)] \qquad [\tfrac{1}{\pi} \text{ from } (1)]$$

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{\text{poles of } f(z)} \text{Res}[f(z)\csc(\pi z)] \qquad (3)$$

which is what we wanted. Notice that we have the factor Res[csc(πz), 0] = $\frac{1}{\pi}$ in the second to last line of the extended equation. From p. 439 and (12), we think this factor can only be applied if f(z) is non-singular at n. If f(z) = $1/(z^2-1)$, for example, we would have to drop the poles {-1, 1} from the range of the sum. Then we presume the sum would read $\sum_{n=-\infty}^{\infty} (-1)^n f(n)$, n $\notin$ {-1, 1}.