Needham, Visual Complex Analysis, Chapter 9, Exercise 10, p. 448
G. Palmer, December 3, 2017, 1:10 pm PT

10 Use (13) to show that $\sum_{n=1}^{\infty}\left(1 \big/ n^4\right) = \left(\pi^4/90\right)$.

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{\text{poles of } f(z)} \text{Res}[f(z)\cot(\pi z)] \qquad \text{(13), p. 441}$$

Use the S contour described in [5], p. 440. Let f(z) = $1 \big/ z^4$. Let g(z) = f(z) cot(πz).

We first find Res[g(z), 0]. The equation for g(z) at the top of p. 439 is based on f(z) = $1 \big/ z^2$. From this we can deduce that f(z) = $1 \big/ z^4$ will result in terms multiplied by $1 \big/ z^2$.

$$\text{g(z)} = \frac{1}{\pi z^5} - \frac{\pi}{3z^3} - \frac{\pi^3}{45z} - \ldots$$

$$\text{Res[g(z), 0]} = -\frac{\pi^3}{45}$$

To use (13) we need to know that |f(z)| < (const.) / $|z|^2$ for sufficiently large |z|.

$$|f(z)| = \left|1 \big/ z^4\right| = 1/\left|z^4\right| = 1/|z|^4 < 1/|z|^2 \text{ for } |z| > 1$$

The inequality is true for f(z). Then

$$\sum_{n=-\infty}^{\infty}\left(\frac{1}{n^4}\right) = -\pi \left[\sum_{\text{poles of } f(z)} \text{Res}[f(z)\cot(\pi z)]\right]$$

We need only the sum $\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$. Since $1 \big/ z^4$ is an even function, we can divide the RHS by 2.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right) = -\frac{\pi}{2} \left[\sum_{\text{poles of } f(z)} \text{Res}[f(z)\cot(\pi z)]\right]$$

f(z) has only one pole (0) of order 4. Hence,

$$-\frac{\pi}{2}\,\text{Res}[f(z)\cot(\pi z), 0] = -\frac{\pi}{2}\left(-\frac{\pi^3}{45}\right) = \frac{\pi^4}{90}$$

This is what we wanted.

To check this work, we can revert to (12) and follow the method used to deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ presented in §§5, p. 439. Our derivation is similar to that using (13), but it doesn't rise to the level of generality of (13). It is only good for f(z) = $1 \big/ z^4$.

To write an equation comparable to that on p. 439, we need to know that $\oint_S g(z)$ dz vanishes as N goes to infinity. We use part of the results from the derivation of (13), i.e. that the perimeter of S = (8N+4). For sufficiently large N, $|\cot(\pi/z)| < 2$ everywhere on S (p. 440). But we note that Needham presented the derivation of $\pi^2/6$ on p. 439 before establishing the inequality. Then, by (5), p. 387,

$$\left|\oint_S g(z)\, \mathrm{dz}\right| \leq (\text{Max } |g| \text{ on S})\ (\text{perimeter of S}) < \frac{2}{N^4}(8\text{N} + 4)$$

The RHS tends to zero as N tends to infinity, so we can use the value for Res[f(z) cot(πz)] in (12), p. 439.

Now we can write

$$(1 / (2\, \pi\, i)) \oint_S \text{g(z) dz} = \text{Res[g(z), 0]} + \sum_{n=-N}^{-1} \text{Res[g(z), n]} + \sum_{n=1}^{N} \text{Res[g(z), n]}$$

We have shown that the LHS vanishes as N tends to infinity.

$$0 = -\frac{\pi^3}{45} + \frac{2}{\pi} \sum_{n=1}^{N} \left(\frac{1}{n^4}\right)$$

$$\sum_{n=1}^{N} \left(\frac{1}{n^4}\right) = \frac{\pi^4}{90}, \text{ where N} = \infty$$

We have checked our result by rederiving it from (12) for f(z) = $1 / z^4$.