Needham, *Visual Complex Analysis*, Chapter 9, Exercise 1
G. Palmer, September 23, 2017, 6 pm PDT

1. If C is the unit circle, show that

$$\int_0^{2\pi} \frac{dt}{1+a^2-2a\cos t} = \oint_C \frac{i\,dz}{(z-a)(az-1)}.$$

Answer: The basic approach for rewriting a real integral as a contour integral is described on p. 437. The contour integral is doubtless easier to solve than the real.

$\cos t = \frac{1}{2}[z + \frac{1}{z}]$, $dt = i\,z\,dz$, $dz = \frac{dt}{iz} = \frac{i\,dt}{-z}$, from p. 437, [4] b, substituting t for θ

$$\int_0^{2\pi} dt/\left(1+a^2-2a\cos t\right) = \oint_C (i\ dz)/\left(-z\left(1+a^2-2a\,\frac{1}{2}\left[z+\frac{1}{z}\right]\right)\right) = \oint_C (i\ dz)/\left(az^2-a^2z - z + a\right)$$

$$= \oint_C \frac{i\ dz}{(z-a)(az-1)}$$

Use Cauchy's Formula to deduce that if 0 < a < 1, then

$$\int_0^{2\pi} \frac{dt}{1+a^2-2a\cos t} = (2\pi)/\left(1-a^2\right).$$

Answer:

$\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}dz = f(a)$ Cauchy's Formula (1), given, p. 427

$$\int_0^{2\pi} \frac{dt}{1+a^2-2a\cos t} = \oint_C \frac{i\ dz}{(z-a)(az-1)}$$

Then

$$\frac{1}{2\pi i}\oint_C \frac{i\ dz}{(z-a)(az-1)} = f(a),\ \textit{where}\ f(z) = \frac{i}{az-1}$$

$$\Longrightarrow \frac{1}{2\pi i}\oint_C \frac{i\ dz}{(z-a)(az-1)} = \frac{i}{a^2-1}$$

$$\oint_C \frac{i\ dz}{(z-a)(az-1)} = \frac{2\pi}{1-a^2}$$

We can see that f(a) has a singularity at a = 1, and since C is a unit circle, the equation is true when a is 0 < a < 1. The equation trivially true for 0 ≤ a < 1. When a = 0, f(0) = $\frac{i}{-1}$ = -i, so $\oint_C \frac{i\ dz}{(z-a)(az-1)}$ = -i (2 π i) = 2π = $\frac{2\pi}{1-a^2}$.