Needham, *Visual Complex Analysis*, Notes on Chapter 8
G. Palmer, May 25, 2018, 2 pm PST

Synopsis with focus on Cauchy's Theorem

On p. 378, Cauchy's Theorem is given as "any two integrals [on contours between the same two points] will agree, provided that the mapping is analytic everywhere in the region lying between the two contours [both from A to B]." This is equivalent to saying that the integral on a closed loop vanishes everywhere in the region where the mapping is analytic. Needham describes this as "that single horn of plenty," implying that more theorems may be derived from it. Notice, that integrals on closed loops containing singularities (points where the mapping is not analytic) may still vanish. But if the loop does not contain a singularity, we know the integral will vanish.

pp. 381-382. In this section, equations are presenting using two letter variables, such as PQ, QR, and "the chord AB". When these variables appear in equations, they actually stand for the lengths of the straight line segments between, for example, points A and B. That is, they represent |B-A|, |P-Q|, etc.

On p. 389 we find that the loop integral of function 1/z fails to vanish on a loop that encloses the origin. It has value 2π i. This does not contradict Cauchy's Theorem because 1/z is not analytic throughout the interior of the loop; it has a singularity at the origin. On p. 390, we find that when the loop does not enclose the origin the integral on 1/z actually does vanish in accordance with Cauchy's Theorem.

On p. 395-396, VI Power Functions. 1 Integration along Circular Arc. We find various power functions integrated on circles containing the origin. We know they contain the origin, because the text refers to contour K in [11] and to [17] a and b, all of which are circles centered on the origin. On p. 396, it is concluded that if m ≠ -1, $R_M = \frac{1}{m+1}\left[B^{m+1} - A^{m+1}\right]$. Needham does not explicitly state it here, but it seems clear that the integral of a power function where m ≠ -1 over an interval with equal endpoints goes to zero even though it is analytic and contains a singularity. This does not contradict Cauchy's theorem.

There are two cases to consider: m > 0 and m < -1 (the zero case is trivial). On p. 397, we find "Take the case n > 0 first. Since z^m is then analytic throughout the plane, it follows directly from Cauchy's theorem that all contours will yield the same value." In other words, the integral vanishes, even when the circle contains the origin. The function z^m, m > 0, is analytic at 0, so Cauchy's Theorem says it will vanish. This is supported by (10). In the case of m < -1, there is a singularity at 0, but the integral vanishes anyway by (10). Cauchy's Theorem does not preclude its vanishing.

P. 399 draws on the fact established by (10) that the integral of z^m vanishes on a circular loop around the origin and concludes that the integral has path independence (except for m = -1). The Deformation Theorem is an application of Cauchy's Theorem, which Needham describes as a "'dynamic' version of it. It is used to verify (12), the formula for the integral of a function with multiple singularities inside a closed loop. The Deformation Theorem leads to the use of residues to calculate integrals, because for functions with powers greater than 0 and less than -1, integrals on closed loops vanish, leaving only the residue integral on components with powers of -1.

End of synopsis

***p. 381**, ¶ 2, regarding Figure [5a], "Note that if we rotate $\tilde{D}\tilde{C}$ about P (keeping the ends glued to the verticals) until it becomes tangent at DC, the area beneath it will remain constant [why?]."*

The rotation creates two new triangles, one below and one above $\tilde{D}\tilde{C}$. The area of triangle $\tilde{D}$PD equals the area of $\tilde{C}$PB. Hence, the area below the trapezoid equals the area of the rectangle from which it was created.

***p. 381**, "The actual area lies between the two values furnished by AB and DC, and ... the error introduced by using either rule cannot exceed the area of the small quadrilateral ABCD, namely [exercise], (PQ) · Δ.*

Rotate AB about Q and DC about P until DC has returned to the position of $\tilde{D}\tilde{C}$, keeping the ends glued to the verticals as before. The area lost and gained to the trapezoid under each rotated line will be equal, so the area will remain the same. Since P is the midpoint of AB, and Q is the midpoint of CD, PQ = (AD + BC)/2 and $\mathcal{A}$(ABCD) = PQ · Δ.

p. 382, trace calculations; $\mathcal{A}$ = area

$PQ \cdot QR = (AQ)^2$ because Q is a midpoint by construction (2)

$PQ \cdot PR = (DP)^2$ substituting into (2) (3)

$PQ \cdot (2/\kappa) \cos\theta = ((1/2)\,\Delta \sec\theta)^2$ substituting into (3)

$PQ = ((1/2)\,\Delta \sec\theta)^2 / (2/\kappa) \cos\theta$

$= (1/8)\,\kappa\,\Delta^2 \sec^3\theta$

$PQ \cdot \Delta = (1/8)\,\kappa\,\Delta^3 \sec^3\theta = \mathcal{A}(ABCD)$

P. 383-387. Notes on the Complex Integral and Complex Riemann Sums

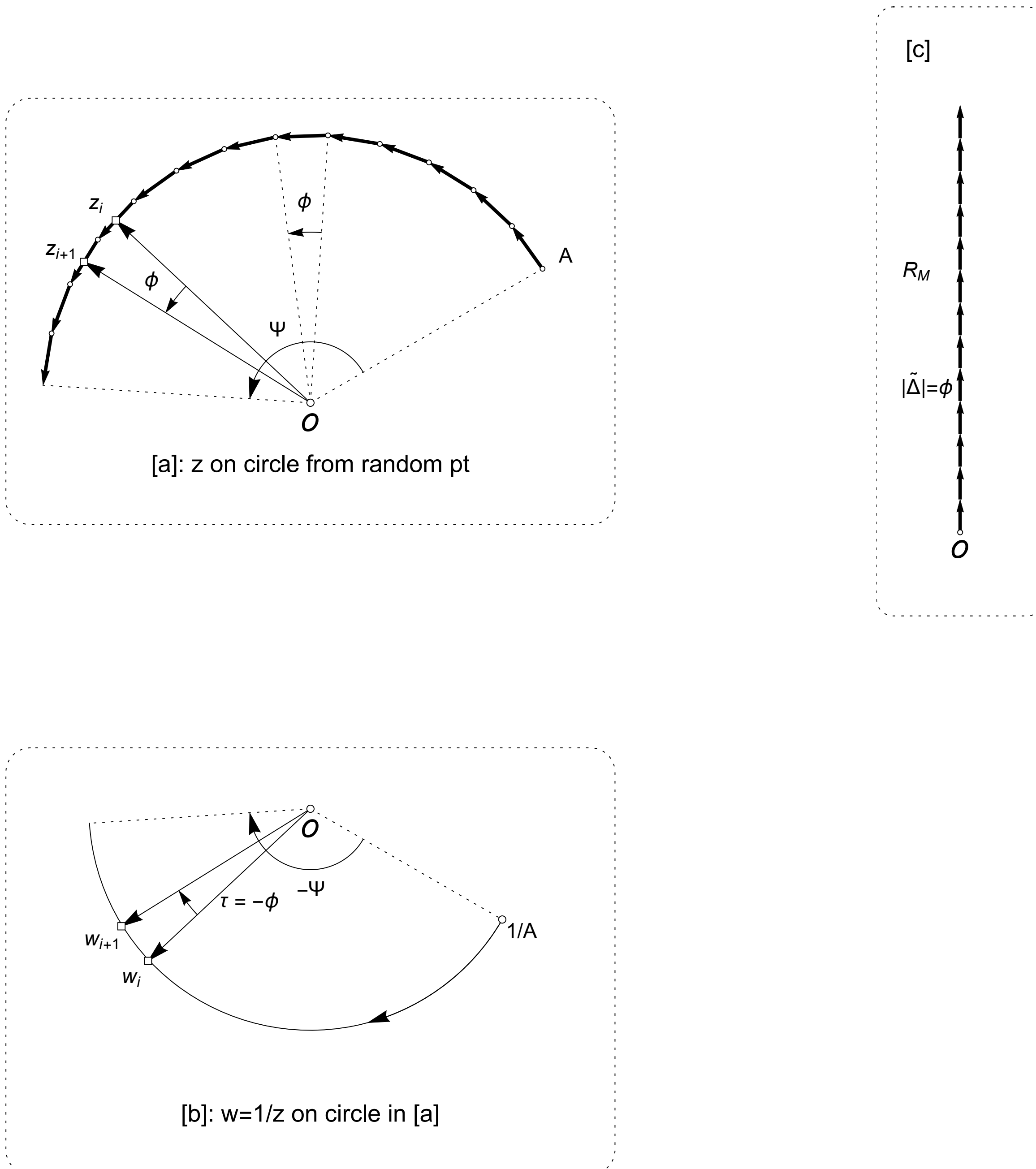
Needham commonly places contours and variables (usually z) on one complex plane and functions (or w) on another. Figures [7], [8], and [9] suggest that we can introduce a third complex plane, the Riemann plane to plot the Riemann integral. Table 1 indicates where the chapter's symbols and symbol combinations for functions, vectors, and angles appear in this framework.

Table 1. Symbols of planes pertaining to Riemann integral

plane	function	Δ vector	arg	initial arg
z	z	Δ_i	ϕ_i	β
w	f(z), w_i	$w_{i+1} - w_i$	τ_i	α
R_M	f(z)·dz	$\tilde{\Delta}_i = w_i \Delta_i$	$\tilde{\phi}_i = \phi_i + \tau_i$	$\alpha + \beta$

P. 389, ¶ 1, "Convince yourself that this formulation of the result remains valid even if K begins at a random point of the circle instead of on the real axis."

Figure 1



Figures 1a, b show that it is possible to start at another place on the circle, have z traverse by intervals of Δ, and plot (1/z) on a circle of radius 1/|A| with intervals of $\tilde{\Delta}$ to an angle -Ψ. Then the $\tilde{\Delta}$ of R_M are obtained by cancelling the rotations of the Δ's but keeping the length. Does the integral equal i Ψ?

$$i\,\Psi = \frac{13\,i\,\pi}{16} = i\,|\,13\,\phi\,|$$

p. 390, Figure [12]

In Figure [12], there is a right angle at the vertex of dr and r dθ. Note that the short segment dr is not a continuation of the ray created by dθ.

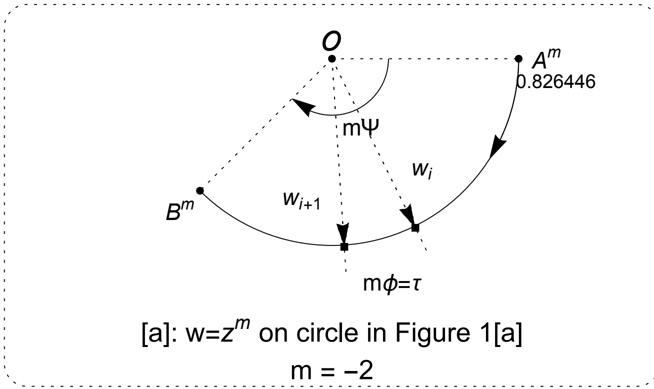
p. 392, ¶ 1 re. (7), *Finally, note that (6) can easily be generalized [exercise] to*

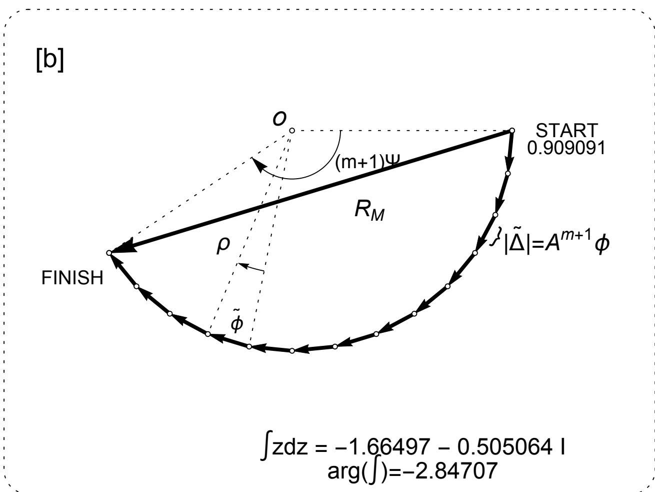
$$\oint_L \frac{1}{z-p} dz = 2\pi\, i\, \nu(L, p).$$

If f = $\frac{1}{z-p}$, then p is a pole in the z-plane. We can draw concentric circles as in [13], p. 391, but centred at p, and make the same argument as was made for 1/z. For p outside a loop L, horizontal components of $\tilde{\Delta}$ cancel and the Riemann sum of vertical components is zero. For p inside a loop L, horizontal components cancel, but vertical components sum to 2π i. Therefore,

$$\oint_L \frac{1}{z-p} \mathrm{dz} = 2\pi\, \mathrm{i}\, \nu(\mathrm{L}, \mathrm{p}).$$

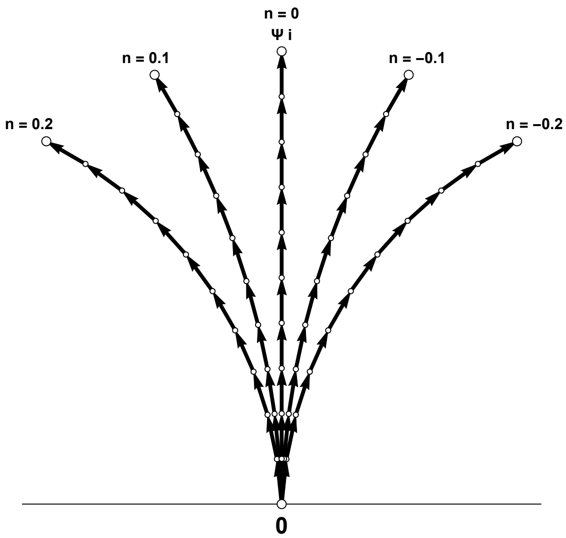
Figure 2. m = −2





p. 396, ¶ -1, *Before continuing you may care to sketch another case for yourself; m = -2, might be a fun one.*

We try m = -2, basing it on the same circle but leaving out the tangents and their turning angles. Figures 2 [a] and [b] show the result. The integral looks similar to that for m = 2, but it’s value and direction are somewhat different.



Needham Figure [18]. ∫ on powers (reproduction)

p. 397, [18]. In Figure [18], we try several powers of z in an attempt to reproduce something like Figure [18], p. 397. It appears that Needham begins with $|\tilde{\Delta}|$i, pointing vertically, and applies the turning angles $\tilde{\phi}$ to each increment starting from the origin. The n's were calculated from n = m + 1. I followed the formulas on p. 395 and 396 using a ϕ of Pi/6 following a suggestion by Vasco.

A = 1 (unit circle radius and starting point for z)

$\phi\tilde{}$ =(m+1)ϕ

$\Delta\tilde{}_0 = A^{m+1}|\phi|$i (an initial vertical Δ)

Now rotate $\Delta\tilde{}_0$ about 0 by the angle $\phi\tilde{}$ times j, its position in the integral, giving it an appropriate multiple of the turning angle. $\phi\tilde{}$ is different for each m. This defines 10 arrows for each m, but doesn't arrange them.

$\tilde{\Delta}_j = \tilde{\Delta}_0\, e$^{$i\,(j\,\phi)$}, j = (1..10)

Now plot each $\tilde{\Delta}_j$ on the sum of all previous $\tilde{\Delta}_j$. The turning angles are easy to calculate: (m+1)ϕ. The difference between n = m+1 of .1 and .2 means that the turning angle is doubled.

p. 400, line 4: *Assuming [as in 20a] that L encloses both singularities, use this formula to verify (12) for this particular function.*

$\oint_L f(z)dz = \oint_J f(z)dz + \oint_K f(z)dz$ (12), p. 399

The function is

$f(z) = \frac{i}{z+i} - \frac{i}{z-i}$ p. 400

Use

$$\oint_C f(z)dz = 2\pi[\nu(C, i) - \nu(C, -i)] \qquad \text{p. 400}$$

The loop C is “any loop”. Let J and K be non-touching circles as illustrated in [20] and integrated in (12), so that J contains -i but not i, and K contains i but not -i. Hence, $\nu[j, -i] = \nu[K, i] = 0$.

$$\oint_J f(z)dz + \oint_K f(z)dz$$

$$= 2\pi[\nu(J, i) - \nu(J, -i)] + 2\pi[\nu(K, i) - \nu(K, -i)]$$

$$= 2\pi[\nu(L, i) - 0] + 2\pi[0 - \nu(L, -i)] \qquad \textit{prop.}\ \nu(L, i) = \nu(J, i)\ \&\ \nu(L,-i) = \nu(K,-i)$$

$$= 2\pi[\text{v}(L,i) - \text{v}(L, -i)]$$

$$= \oint_L f(z)dz$$

This verifies (12) for this function.

p. 410, ¶ 1, Use parametric evaluation to show that $\oint_L \bar{z}^2 dz = 4\,\pi\, i\, \bar{q}\, \rho^2$.

$$z(t) = q + \rho e^{it}; \quad v(t) = dz/dt = \rho i e^{it}$$

$$\bar{z}(t) = \bar{q} + \rho e^{-it}$$

$$\bar{z}(t)^2 = \bar{q}^2 + 2\,\bar{q}\,\rho e^{-it} + \rho^2\, e^{-2\,it}$$

$$\bar{z}(t)^2 v(t) = \rho\, i\, [\bar{q}^2\, e^{it} + 2\bar{q}\rho + \rho^2\, e^{-it}]$$

$$\oint_L \bar{z}^2 dz = \int_0^{2\pi} \bar{z}(t)^2\, v(t)\, dt \ = \ \rho\, i \int_0^{2\pi} \left[\bar{q}^2\, e^{it} + 2\,\bar{q}\,\rho + \rho^2\, e^{-it}\right] dt$$

$$= \rho\, i\, [\int_0^{2\,\pi} \bar{q}^2\, e^{it} dt + \int_0^{2\pi} 2\,\bar{q}\,\rho\, dt + \int_0^{2\pi} \rho^2\, e^{-it}\, dt]$$

$$= \rho\, i[\bar{q}^2\, \frac{e^{it}}{i} + 2\, \bar{q}\, \rho t - \rho^2\, \frac{e^{-it}}{i}]_0^{2\pi}$$

$$= \rho\, i\, [2\, \bar{q}\, \rho\, 2\,\pi - 0] = 4\,\pi\, i\, \bar{q}\, \rho^2$$

p. 410, ¶ 2, *By way of contrast with the non-analytic examples above, and as further practice with this method, confirm (using the same contour L* [“the circle”]*) that $\int_L z^2\, dz = 0$, as predicted by either Cauchy’s Theorem or the Fundamental Theorem.*

z^2 is analytic throughout the disc of radius ρ, so by Cauchy’s Theorem, the integral disappears.

Vasco comments in personal communication:

“Since z^2 is analytic it has an antiderivative and because we are talking about a closed loop L the Fundamental Theorem must give us the result zero.

“More explicitly

$$\oint_L z^2 \mathrm{dz} = \int_p^p z^2 \mathrm{~dz} = \left[\frac{z^3}{3}\right]_p^p = 0 \text{ where p is any point on the circle L.}$$

“The above is true for any closed loop L, not just a circle.”

By (10), the equation for power functions,

$$\int_L z^2\, dz = \frac{1}{3}\left[B^3 - A^3\right] = 0, \text{ because B = A = 1}$$

By parametric evaluation

$$\mathrm{z(t) = q} + \rho \mathrm{e}^{\mathrm{it}}$$

$$\mathrm{v(t) = i}\rho \mathrm{e}^{\mathrm{it}}$$

$$z(t)^2 = q^2 + 2\,\mathrm{q}\rho \mathrm{e}^{\mathrm{it}} + \rho^2\, e^{2\,\mathrm{it}}$$

$$\int_0^{2\pi} z(t)^2\, v(t)\, dt = \int_0^{2\pi}\left(q^2 + 2\, q\, \rho \mathrm{e}^{\mathrm{it}} + \rho^2\, e^{2\,\mathrm{it}}\right) \mathrm{i}\rho \mathrm{e}^{\mathrm{it}}\, dt$$

$$= \mathrm{i}\rho \int_0^{2\pi}\left(q^2 + 2\, q\, \rho \mathrm{e}^{\mathrm{it}} + \rho^2\, e^{2\,\mathrm{it}}\right) e^{\mathrm{it}}\, dt$$

$$= \mathrm{i}\rho^3 \int_0^{2\pi} e^{\mathrm{it}}\, dt + 2\,\mathrm{iq}\rho^2 \int_0^{2\pi} e^{2\,\mathrm{it}}\, dt + \mathrm{i}\rho^3 \int_0^{2\pi} e^{3\,\mathrm{it}}\, dt$$

$$= \rho^3 e^{\mathrm{it}} + \mathrm{q}\rho^2 e^{2\,\mathrm{it}} + \rho^3 \frac{e^{3\,\mathrm{it}}}{3} \Big|_0^{2\pi} = \left(\rho^3 + \mathrm{q}\rho^2 + \rho^3\right) - \left(\rho^3 + \mathrm{q}\rho^2 + \rho^3\right) = 0$$

p. 410, ¶ 2, *Confirm that* $\int_E z dz = 0$, *when E is a origin-centred ellipse*.

$$\mathrm{z(t) = pe}^{\mathrm{it}} + \mathrm{qe}^{-\mathrm{it}}, \qquad \mathrm{v(t) = pie}^{\mathrm{it}} - \mathrm{iqe}^{-\mathrm{it}}$$

$$\mathrm{z(t)\ v(t)} = p^2\, \mathrm{ie}^{2\,\mathrm{it}} - \mathrm{pqi} + \mathrm{pqi} - q^2 \mathrm{ie}^{-2\,\mathrm{it}}$$

$$= p^2\, \mathrm{ie}^{2\,\mathrm{it}} - q^2 \mathrm{ie}^{-2\,\mathrm{it}}$$

$$\int_E z dz = \int_0^{2\pi} z(t)\, v(t)\, dt = \mathrm{i}\int_0^{2\pi}\left(p^2\, e^{2\,\mathrm{it}} - q^2\, e^{-2\,\mathrm{it}}\right) dt$$

$$= \mathrm{i}\left[\frac{p^2\, e^{2\,\mathrm{it}}}{2\, i} + \frac{q^2\, e^{-2\,\mathrm{it}}}{+2\, i}\right]_0^{2\pi} = \left[\frac{p^2\, e^{2\,\mathrm{it}}}{2} + \frac{q^2\, e^{-2\,\mathrm{it}}}{+2}\right]_0^{2\pi}$$

$$= \frac{p^2+q^2}{2} - \frac{p^2+q^2}{2} = 0$$

p. 410, ¶ 3, *take the contour to be a section of the parabola $y = x^2$ between 0 and 1 + i; in temporal terms, this can be represented as $z(t) = t + it^2$ ($0 \leq t \leq 1$). Integrate z along this contour, first using the Fundamental Theorem, then parametrically.*

Fundamental Theorem: We evaluate F(x) at the beginning and end points F(1) - F(0).

$$\int_0^{1+i} z dz = \frac{z^2}{2}\Big|_0^{1+i} = (1+i)^2/2 = (1 + 2i - 1)/2 = i$$

Parametric Evaluation

$$z(t) = t + it^2$$

$$v(t) = 1 + 2it$$

$$\int_0^1 z(t)\, dz = \int_0^1 \left(t + it^2\right)(1 + 2\,it)\, dt$$

$$= \int_0^1 \left(t + 3\,it^2 - 2\,t^3\right) dt$$

$$= \int_0^1 t\, dt + 3\, i \int_0^1 t^2\, dt \; -2 \int_0^1 t^3\, dt$$

$$= \left[\frac{t^2}{2} + it^3 - \frac{t^4}{2}\right]_0^1$$

$$= \frac{1}{2} + i - \frac{1}{2} = i$$

The parametric equation in t is identical in form to the equation z(x) = x + if(x), where x represents time, in Exercise 1.

***p. 410, ¶ 3**, use the Fundamental Theorem to evaluate the integral for e^z.*

We evaluate e^z on the parabola between 0 and 1 + i.

$$\int_0^{1+i} e^z\, dz = e^{1+i} - e^0 = e^{1+i} - 1$$

$$= ee^i - 1 = e(\cos 1 + i \sin 1) - 1$$

$$= (e \cos 1 - 1) + i\, e \sin 1$$

The imaginary part of this is (e sin 1).

Parametric evaluation:

$$z(t) = t + it^2$$

$$v(t) = (1 + 2it)\, dt$$

$e^z = e^{t+it^2}$

$e^z\, v(t) = e^{t+it^2}(1+2it)\,dt$

$\int_0^1 e^z\, dz = \int_0^1 e^{t+it^2}(1+2\,it)\,dt$

$= \int_0^1 e^{t+it^2}\,dt + \int_0^1 2\,it\,e^{t+it^2}\,dt$

$= \int_0^1 2\,it\,e^t\,e^{it^2}\,dt + \int_0^1 e^t\,e^{it^2}\,dt$ (cosmetic reversal of order of terms)

$= \int_0^1 2\,i\,t\,e^t\left(\cos t^2 + i\sin t^2\right)dt + \int_0^1 e^t\left(\cos t^2 + i\sin t^2\right)dt$ (Euler)

$\text{Im}\int_0^1 e^z\, dz = \int_0^1 2\,t\,e^t\cos t^2\,dt + \int_0^1 e^t\sin t^2\,dt$

$= \int_0^1\left(2\,t\cos t^2\,dt + \sin t^2\right)e^t\,dt$ (sum of integrals is integral of sum)

By equating the imaginary part of your answer with the imaginary part of the parametric evaluation, deduce that

$\int_0^1\left(2\,t\cos t^2 + \sin t^2\right)e^t\,dt = e\sin 1$

The deduction is confirmed.

p. 412, Subsection 2 The Explanation, lines 5-7: “the integral along the bottom edge of this square can be approximated by the single term A ϵ: the image of the midpoint a, times the number along this edge.”

The integral of the bottom side of the left hand square can be found by multiplying its number or complex vector ϵ by the image of its midpoint “a”. This image is the “A” of the square on the right hand side. “A” is an amplification and twist of “a”. The operation is equivalent to wΔ, f(a)ϵ, or $\int_0^\epsilon A\,dz = A\epsilon$.