Needham, Chapter 8, Exercise 9, p. 421.
G. Palmer, July 5, 2017, 6:30 am PST

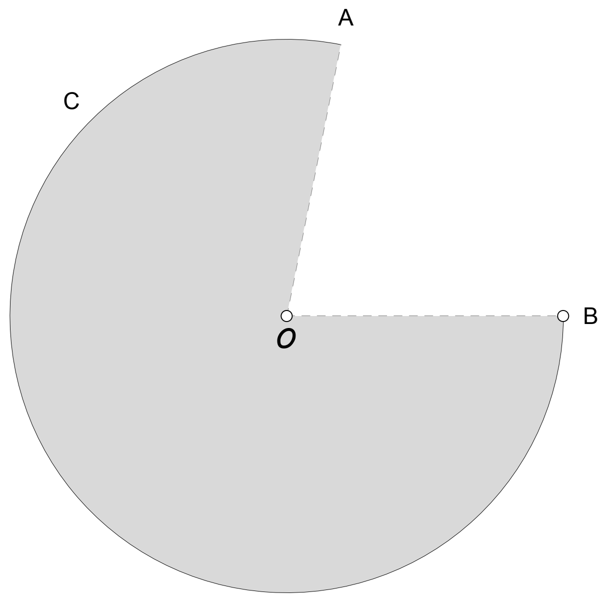

Figure 1: Area swept out by $\overline{z}$ on open circle
origin inside curvature

9 What is the generalization of (8) to the case where the contour is not closed?

(8) is an equation involving the area enclosed. It could be generalized to an unclosed area by substituting area swept out by the conjugate traversing the contour, or as Needham puts it "the signed areas subtended by Δ's at the origin". We use "C" rather than "L" because the contour is open.

$\int_C \overline{z}dz$ = 2i (the signed areas subtended by Δ's at the origin)

We could also describe the signed area as "enclosed", which is to be read more generally as "area swept out and partially enclosed by rays from the origin as z traverses from A to B". We give two examples.

In Figure 1, there are no singularities nor self-intersections inside this simple contour. Lack of closure means we can not apply (9) $\oint_L \overline{z}dz = 2i \sum_{\text{inside}} v_j \mathcal{A}_j$. Notice in Figure 1 that the rays (dashed lines) from the origin to A and B enclose the area ABO, where AB is the circular segment. We can write the integral involving this signed area subtended by Δ's as z traverses A to B.

$\int_C \overline{z}dz = \int_A^B \overline{z}dz$ = 2i (signed area subtended by Δ's at the origin)

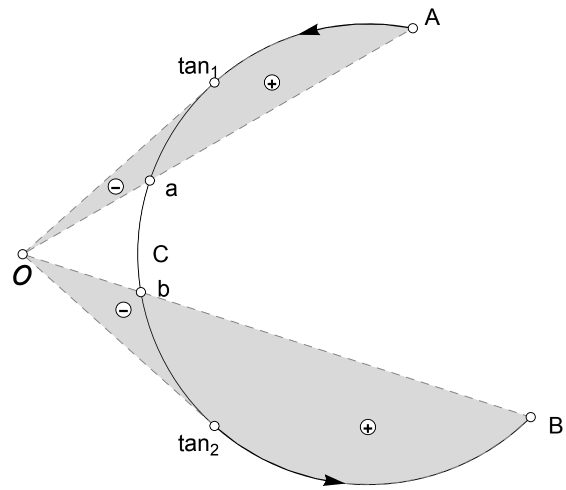

Figure 2: Area swept out by $\overline{z}$ on open circle
origin outside curvature

In Figure 2, rays from the origin to points on contour C have two points of intersection with C, except for rays to the tangent points $\tan_1$ and $\tan_2$, which have only a single point each where they touch C. The traversal of z on C has positive rotation around the origin from A to $\tan_1$ and from $\tan_2$ to B. It has negative rotation around the origin between $\tan_1$ and $\tan_2$. We assume we are only interested in the area under the chords a-A and b-B, that is, the area subtended by Δ's from a to A and from b to B. On the far side of the contour from the origin, Δ carries z counterclockwise, yielding a positive element of area, and on the near side Δ carries z clockwise, yielding a negative element of area. The result is that the area outside (see minus signs) cancels, so we can write:

$$\int_C \overline{z}dz = \int_A^{\tan_1} \overline{z}dz + \int_{\tan_2}^B \overline{z}dz$$

$$= 2i \text{ (signed areas subtended by Δ's)}$$

For different purposes, we might wish to add or subtract the area swept out between the origin and C in the shape aOb. This term would take the form $(-\int_a^b \overline{z}dz)$ = 2i (area subtended by Δ's in segment a to b). Strictly interpreted, the area associated with C in Figure 2 would include this term.

Our generalization is the following:

$$\int_C \overline{z}dz = 2i\text{(area subtended by Δ's).}$$