Needham, Chapter 8, Exercise 8, p. 421.
G. Palmer, June 21, 2017, 4:25 pm PST

8 The figure below [not included] *shows four simple loops, and in each case we have indicated how much shaded area is inclosed. Use parametric evaluation to verify equation (8) for each of the four loops.*

$$\oint_L \overline{z}\, dz = 2i \text{ (area enclosed)} \qquad\qquad (8), \text{ p. } 394$$

We rewrite (8)

$$\mathcal{A} = \frac{1}{2i} \oint_L \overline{z}\, dz$$

(i) Loop (i) is a circle of radius R and area πr^2. We rewrite (8).

$$\mathcal{A}(C) = \frac{1}{2i} \int_0^{2\pi} \overline{z}(t)\, v(t)\, dt$$

$$z(t) = Re^{it}, \qquad \overline{z}(t) = Re^{-it}, \qquad v(t) = iRe^{it}$$

$$\mathcal{A}(C) = \frac{1}{2i} \int_0^{2\pi} Re^{-it}\, iRe^{it}\, dt$$

$$= \frac{R^2}{2} \int_0^{2\pi} dt = \frac{R^2}{2} t \,\Big|_0^{2\pi} = \frac{R^2}{2}(2\pi) = \pi R^2$$

Parametric evaluation verifies (8).

(ii) Loop (ii) is an ellipse with major axis a, minor axis b and area of π ab.

$$z(t) = a \cos t + i\, b \sin t$$

$$\overline{z}(t) = a \cos t - i\, b \sin t$$

$$v(t) = -a \sin t + i\, b \cos t$$

$$\mathcal{A}(E) = \frac{1}{2i} \int_0^{2\pi} \overline{z}(t)\, v(t)\, dt$$

$$= \frac{1}{2i} \int_0^{2\pi} (a \cos t - i\, b \sin t)\, (-a \sin t + i\, b \cos t)\, dt$$

$$= \frac{1}{2i} \int_0^{2\pi} \left(-a^2 \cos t \sin t + i\, \mathrm{ab} \cos^2 t + i\, \mathrm{ab} \sin^2 t + b^2 \cos t \sin t\right) dt$$

$$= \frac{1}{2i} \int_0^{2\pi} \left[\left(b^2 - a^2\right) \cos t \sin t + i\, \mathrm{ab} \left(\cos^2 t + \sin^2 t\right)\right] dt$$

$$= \frac{1}{2i} \left[\left(a^2 - b^2\right) \frac{1}{2} \cos^2 t + i\, \mathrm{abt}\right]_0^{2\pi}$$

$$= \frac{1}{2i}\left[\left(a^2 - b^2\right) \frac{1}{2} (1 - 1) + i\, \mathrm{ab}(2\pi - 0)\right] = \pi\, \mathrm{ab}$$

Parametric evaluation verifies (8).

(iii) Figure (iii) appears to be an enclosed quarter disc with radius R and area $\pi r^2/4$. We use the integral from (i) with a range of 0 to $\pi/2$ on the quarter circle and then we integrate on the sides somewhat as we did in for the loop integral exercise 5.

$$\mathcal{A}(\text{quarter disc}) = \oint_L = \oint_{(0,\pi/4)} + \oint_{\text{iR}} + \oint_R$$

$\oint_L$ on the quarter circle:

$$\oint_{L(0,\pi/4)} = \frac{R^2}{2} t \,\Big|_0^{\pi/2} = \frac{\pi R^2}{4}$$

$\oint_L$ on the side 0 to iR:

$$z(t) = it, \qquad \bar{z}(t) = -it, \qquad v(t) = i$$

$$\mathcal{A}_{\text{iR}} = \frac{1}{2i} \oint_{L(0,\text{iRt})} \bar{z}(t)\, v(t)\, dt$$

$$\frac{1}{2i}\int_0^R \bar{z}(t)\ v(t)\ dt = \int_0^R (-it)\, i\, dt = \int_0^R t\, dt$$

$$= \frac{1}{2i} \frac{t^2}{2} \,\Big|_0^R = \frac{R^2}{4i}$$

But, since we actually need a negative (counterclockwise) direction of integration, this should be $-\frac{R^2}{4i}$, so $\mathcal{A}_{\text{iR}} = -\frac{R^2}{4i}$.

$\oint_L$ on the side 0 to R:

$$z(t) = t, \qquad \bar{z}(t) = t, \qquad v(t) = 1$$

$$\mathcal{A}_R = \frac{1}{2i} \int_0^R t\, dt = \frac{1}{2i} \frac{t^2}{2} \,\Big|_0^R = \frac{R^2}{4i}$$

Then

$$\mathcal{A} = \oint_L = \oint_{(0,\pi/4)} + \oint_{(0,\text{iR})} + \oint_{(0,\,R)} = \frac{\pi R^2}{4} - \frac{R^2}{4i} + \frac{R^2}{4i} = \frac{\pi R^2}{4}$$

Parametric evaluation verifies (8).

(iv) Figure (iv) is a triangle with vertices at R, R+iR, and iR with area $R^2/2$.

We use parametric evaluation of the area on the three sides and add the results. We integrate with t=0 to t=1 in order to complete an interval that is 100 percent of the complex number over which we are integrating.

Side R to (R+iR):

$z(t) = R + iRt, \qquad \overline{z}(t) = R - iRt, \qquad v(t) = iR$

$\frac{1}{2i}\int_0^1 \overline{z}(t)\, v(t)\, dt = \frac{1}{2i}\int_0^1 (R - iRt)\, iR\, dt$

$= \frac{1}{2i}\int_0^1 \left(iR^2 + R^2 t\right) dt$

$= \frac{R^2}{2i}\left[\int_0^1 i\, dt + \int_0^1 t\, dt\right]$

$= \frac{R^2}{2} t \,\Big|_0^1 + \frac{t^2}{4i}\,\Big|_0^1 = \frac{R^2}{2} + \frac{R^2}{4i}$

Side (R+iR) to iR:

$z(t) = R+iR -Rt,$
$\overline{z}(t) = R-iR -Rt = R(1-i-t)$
$v(t) = -R$

$i(2-t)+t = 2i -it + t$

$\frac{1}{2i}\int_0^1 \overline{z}(t)\, v(t)\, dt = \frac{1}{2i}\int_0^1 (R(1 - i - t)\,(-R))\, dt$

$= \frac{R^2}{2i}\int_0^1 (i - 1 + t)\, dt = \frac{R^2}{2i}\left[\int_0^1 i\, dt - \int_0^1 dt + \int_0^1 t\, dt\right]$

$= \frac{R^2}{2i}\left[it - t + \frac{t^2}{2}\right]_0^1 = \frac{R^2}{2i}\left[i - 1 + \frac{1}{2}\right]$

$= \frac{R^2}{2} - \frac{R^2}{4i}$

Side iR to R (the hypoteneuse):

$z(t) = iR + (R-iR)\, t$

$\overline{z}(t) = -iR + (R + iR)t = R(-i + t + it)$

$v(t) = (R - iR)$

$\frac{1}{2i}\int_0^1 R(-i + t + it)\,(R - iR)\, dt$

$$\frac{R^2}{2i}\int_0^1 (-i + t + \text{it})(1 - i)\,dt = \frac{R^2}{2i}\int_0^1 (-1 - i + 2t)\,dt$$

$$= \frac{R^2}{2i}\,[\text{-t -it} + t^2]_0^1 = \frac{R^2}{2i}[\text{-1-i+1}] = -\frac{R^2}{2}$$

Then we can add the three sides:

$$\left(\frac{R^2}{2} + \frac{R^2}{4i}\right) + \left(\frac{R^2}{2} - \frac{R^2}{4i}\right) - \frac{R^2}{2} = R^2 - \frac{R^2}{2} = \frac{R^2}{2}$$

Equation (8) is confirmed.