Needham, Chapter 8, Exercise 7, p. 421.
G. Palmer, June 21, 2017, 8:40 pm PST

7 Hold a coin (of radius A) down on a flat surfacce and roll another one (of radius B) round it. The path traced by a point on the rim of the rolling coin is called an epicycloid, and it is a closed curve if A = nB, where n is an integer.

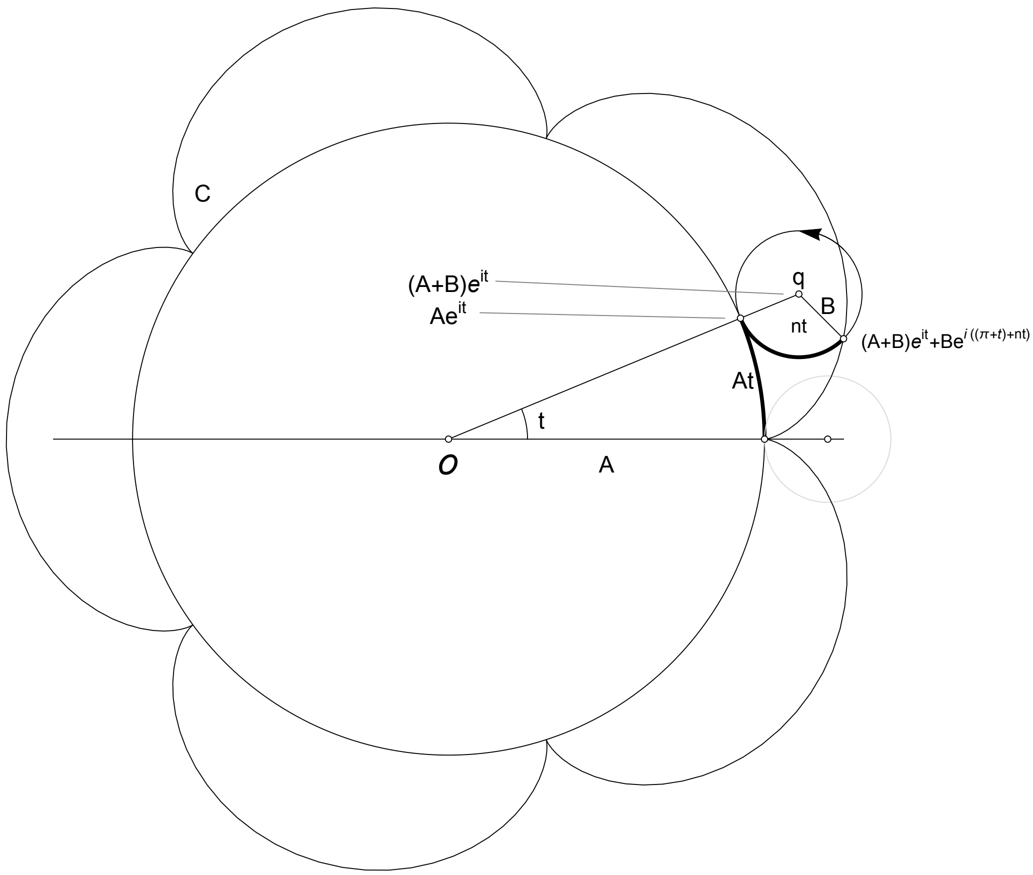

Figure 1: Epicycloid z(t) = B [(n+1)e^{it}−$e^{i(n+1)t}$]

(i) With the centre of the fixed coin at the origin, show that the epicycloid can be represented parametrically as

$$z(t) = B[(n+1)\,e^{it} - e^{i(n+1)t}].$$

Figure 1 shows that the small coin rolls to a point of contact at A to a new point of contact at Ae^{it}. The initial point of contact on the small coin rotates by nt about its centre. The centre of the small coin rotates about the origin from (A+B) to $(A + B)\,e^{it}$. The ray from the centre of the small coin to the initial point of contact rotates by (t + nt) to the angle ((π + t) + nt), so the small ray can be described by the vector $Be^{i(\pi + t + nt)} = -Be^{i(n+1)t}$ at q. The tip of this vector describes the curve at time t: $(A + B)\,e^{it}$

$-\mathrm{B}e^{i(n+1)t}$. It can be rewritten as

$$z(t) = (nB + B)\, e^{it} - B\, e^{i(n+1)t} = B[(n+1)\, e^{it} - e^{i(n+1)t}]$$

The epicycloid in Figure 1 was created with this equation.

(ii) By evaluating the integral in (8) parametrically, show that

area of epicycloid = $\pi B^2\,(n+1)\,(n+2)$

From p. 394, we have

$$\oint_L \bar{z}\, dz = 2i\,(\text{area enclosed}) \qquad (8)$$

Then (area enclosed) = $\mathcal{A}$(epicycloid) = $\frac{1}{2i}\int_0^{2\pi} \bar{z}(t)\, v(t)\, dt$

$$\bar{z}(t) = \overline{B\left[(n+1)\, e^{it} - e^{i(n+1)t}\right]}$$

$$= B[(n+1)\, e^{-it} - e^{-i(n+1)t}]$$

$$v(t) = \frac{dz(t)}{dt} = B[i\,(n+1)\, e^{it} - i\,(n+1)\, e^{i(n+1)t}]$$

$$= B(n+1)i\,[e^{it} - e^{i(n+1)t}]$$

$$\frac{1}{2i}\int_0^{2\pi} \bar{z}(t)\, v(t)\, dt = \frac{1}{2i}\int_0^{2\pi} B\left[(n+1)\, e^{-it} - e^{-i(n+1)t}\right] B(n+1)\, i\left[e^{it} - e^{i(n+1)t}\right] dt$$

$= \frac{1}{2}B^2(n+1)\int_0^{2\pi}\left[(n+1)\, e^{-it} - e^{-i(n+1)t}\right]\left[e^{it} - e^{i(n+1)t}\right] dt$ [by removing B^2(n+1)i from integrand]

Notice that we now have a coefficient $\frac{1}{2}B^2$(n+1), which we will use eventually on the evaluated integral.

We see that in order to obtain $\pi B^2\,(n+1)\,(n+2)$, the integral $\int_0^{2\pi}\left[(n+1)\, e^{-it} - e^{-i(n+1)t}\right]\left[e^{it} - e^{i(n+1)t}\right] dt$ must evaluate to 2π (n+2). Expand the integrand.

$$\left[(n+1)\, e^{-it} - e^{-i(n+1)t}\right]\left[e^{it} - e^{i(n+1)t}\right] = (n+1) - (n+1)\, e^{-it}e^{i(n+1)t} - e^{-i(n+1)t}e^{it} + 1$$

$$= (n+2) - (n+1)e^{int} - e^{-int}$$

We appear to have what we want if we take the integral of the first summand.

$$\int_0^{2\pi} (n+2)\, dt = 2\pi\,(n+2)$$

We would like $\left(-(n+1)\int_0^{2\pi} e^{\text{int}}\,dt - \int_0^{2\pi} e^{-\text{int}}\,dt\right)$ to evaluate to 0. We evaluate the integrals of the two negative summands.

$$-(n+1)\int_0^{2\pi} e^{\text{int}}\,dt = -(\text{n}+1)\ \frac{e^{\text{int}}}{\text{in}}\Big|_0^{2\pi} = \frac{-(n+1)}{\text{in}} - \frac{-(n+1)}{\text{in}} = 0$$

$$-\int_0^{2\pi} e^{-\text{int}}\,dt = \frac{-e^{-\text{int}}}{-\text{in}}\Big|_0^{2\pi} = \frac{1}{\text{in}} - \frac{1}{\text{in}} = 0$$

Adding the two gives 0.

Our integral now has the value of 2π (n+2). We multiply by our calculated coefficient $\frac{1}{2}B^2(\text{n}+1)$.

$$\mathcal{A}(\text{epicycloid}) = \frac{1}{2i}\int_0^{2\pi}\overline{z}(t)\,v(t)\,dt = \frac{1}{2}B^2(\text{n}+1)\ 2\pi\ (\text{n}+2) = \pi\, B^2(\text{n}+1)(\text{n}+2)$$

QED