Needham, Chapter 8, Exercise 5, p. 420.
G. Palmer, June 19, 2017, 11:45 pm PST

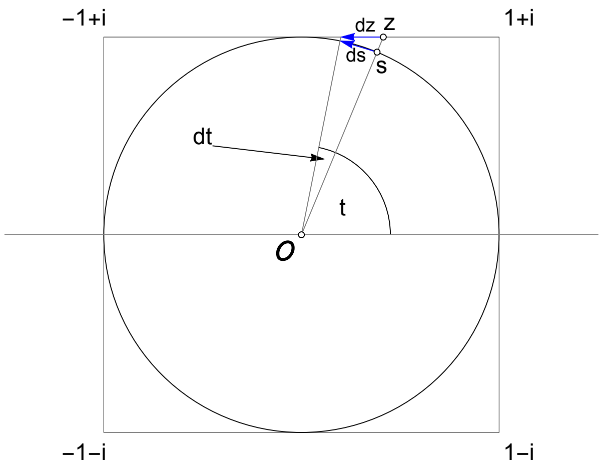

Figure 1: Contour integral of square

5 Evaluate parametrically the integral of (1/z) round the square with vertices ±1±i, and confirm that the answer is indeed 2π i.

We follow the directions of applying the integral of (1/z) round the square, moving always in the counter clockwise direction. We integrate first on the side defined by line segment from (1 - i) to (1 + i). The equation is z(t) = (1 - i) + i t. Then v(t) = i . Parameter t varies from 1-i to 1+i.

Side 1 (right side)

$z(t) = 1 + i\,t$

$v(t) = i$

$$\oint_{rt} (1/z(t))\, v(t)\, dt = \int_{-1}^{1} \frac{i}{1+it}\, dt = \int_{-1}^{1} \frac{i(1-it)}{(1+it)(1-it)}\, dt = \int_{-1}^{1} \frac{i+t}{1+t^2}\, dt$$

$$= i \int_{-1}^{1} \frac{1}{1+t^2}\, dt + \int_{-1}^{1} \frac{t}{1+t^2}\, dt = i \arctan(t) \,|\, {}_{-1}^{1} + (1/2)\, \mathrm{Log}(1+t^2) \,|\, {}_{-1}^{1}$$

$$= \frac{\pi i}{2} + (1/2)\,(\mathrm{Log}\,(2) - \mathrm{Log}(2)) = \frac{\pi i}{2}$$

Side 2 (top side)

z(t) = i - t

v(t) = 1

$\int_{1}^{-1} \frac{1}{i-t}\, dt = \int_{-1}^{1} \frac{i+t}{1+t^2}\, dt$

This is the same integral as for side 1, so $\oint_{\text{top}} = \frac{\pi i}{2}$.

Side 3 (left side)

z(t) = -1 + it

v(t) = i

$\int_{-1}^{1} \frac{i}{-1+\text{it}}\, dt = -\int_{-1}^{1} \frac{i}{(1-\text{it})}\, dt = -\int_{-1}^{1} \frac{i(1+\text{it})}{(1-\text{it})\,(1+\text{it})}\, dt = -\int_{-1}^{1} \frac{i+t}{1+t^2}\, dt$

Then this is the negative of the integral for the right and top sides. It is negative because we integrated from -1 to 1 in the clockwise direction of rotation instead of the reverse, but knowing this, we reverse the sign.

$= -i\, \frac{\pi}{2}$

Since we are only interested in positive values, $\oint_{\text{left}} = \frac{\pi i}{2}$

Side 4 (bottom side)

z(t) = -i + t

v(t) = 1

$\int_{-1}^{1} \frac{1}{-i+t}\, dt = \int_{-1}^{1} \frac{i+t}{1+t^2}\, dt$

This is the same integral as the previous three, so $\oint_{\text{bottom}} = \frac{\pi i}{2}$.

Then the four sides have equal integrals of $\frac{\pi i}{2}$, so $\oint_{\blacksquare} = 2\pi$ i.

An alternative demonstration. See Figure 1.

Inscribe the unit circle inside the square. Let ds = r dt = dt. As dt is allowed to go to 0, dz ⟶ ds. Then,

$i \oint_{C} (1/z)\, dz = i \int_{0}^{2\pi} dt = i\, t \,\Big|_{0}^{2\pi} = 2\pi\, i$