Needham, Chapter 8, Exercise 4, p. 420.
G. Palmer, June 20, 2017, 3:20 PST

4 Write down the values of $\oint_C (1/z)\,dz$ for each of the following choices of C. Then confirm the answers the hard way, using parametric evaluation.

(i) |z| = 1

Parametric evaluation following method in § IX, p. 409:

As z traverses C, (1/z) the winding number is v[C, 0] = 1.

$$\oint_C (1/z)\,dz = 2\,\pi\,i\ v[C, 0] = 2\pi\,i$$

Let z(t) = e^{it}. Then v(t) = ie^{it}dt.

$$\oint_C (1/z)\,dz = \int_0^{2\pi} \frac{v(t)}{z(t)}dt = \int_0^{2\pi} \frac{1}{e^{it}}\,ie^{it}\,dt\ = i\int_0^{2\pi} dt = it\,|_0^{2\pi} = 2\pi\,i$$

Parametric evaluation beginning with the "parametric description of the ordinary graph, y = f(x)", Exercise 1, p. 420.

Let z(x) = x + i f(x) = x + i y(x)

$$y = \sqrt{1-x^2} = (1 - x^2)^{1/2}$$

$$z(x) = x + i\,(1 - x^2)^{1/2}$$

$$dz = (1 + i\,(1/2)(1 - x^2)^{-1/2}\,(-2x))\,dx = (1 - i\,\frac{x}{(1-x^2)^{1/2}})dx$$

$$x(t) = \cos t$$

$$dx = -\sin(t)\,dt$$

$$dz = (-\sin(t) + i\,\frac{\cos(t)\sin(t)}{(1-\cos^2(t))^{1/2}})\,dt = (-\sin(t) + i\,\frac{\cos(t)\sin(t)}{(\sin^2(t))^{1/2}})dt = -\sin(t) + i\cos(t)\,dt = \frac{-i\sin(t) - \cos(t))\,dt}{i}$$

$$= \frac{-e^{it}\,dt}{i} = ie^{it}\,dt$$

$$1/z = \frac{1}{x+i\,(1-x^2)^{1/2}} = \frac{1}{\cos(t)+i(1-\cos(t)^2)^{1/2}} = \frac{1}{\cos(t)+i\sin(t)} = \frac{1}{e^{it}}$$

The problem has now been reduced to the parametric evaluation following method in § IX, p. 409. It is clear that the evaluation beginning with z(t) and v(t) is more economical with the description of the function in terms of the absolute value of z.

$$\oint_C (1/z)dz = \int_0^{2\pi} (1/z(t))\, v(t)\, dt = \int_0^{2\pi} \frac{1}{e^{it}} (i\, e^{it})\, dt = i \int_0^{2\pi} dt = 2\pi\, i$$

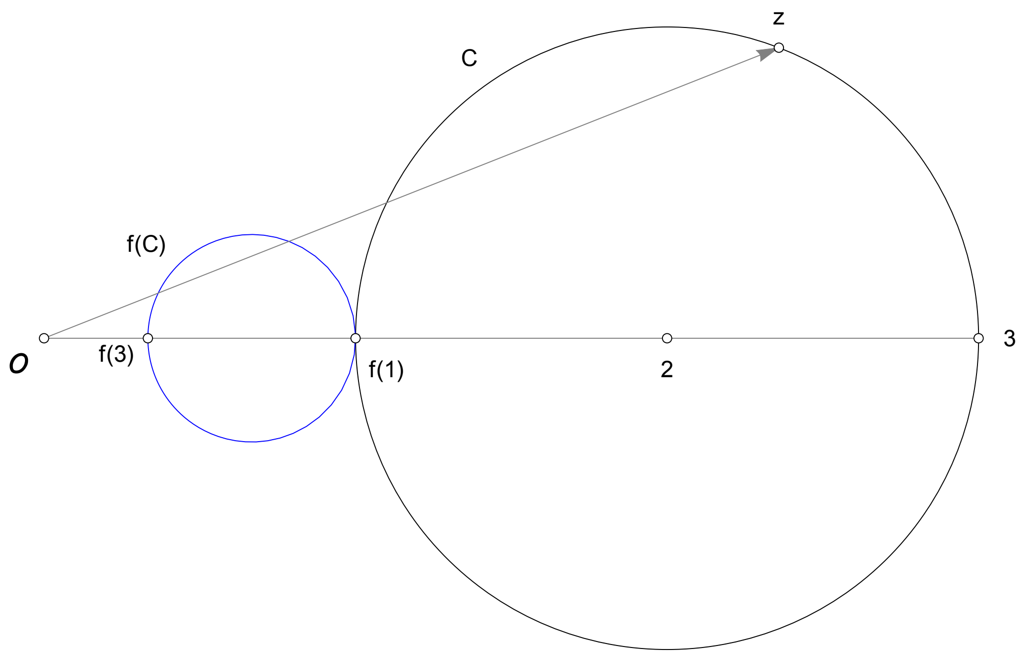

Figure 1: |z–2| = 1, f(z) = 1/z

(ii) |z-2| = 1

C is centred at p = 2. Figure 1 shows that a traversal of z does not encircle the origin. Therefore,

$$v(C, 0) = 0$$

From (6),

$$\oint_C (1/z)\, dz = 2\, \pi\, i\, v(C, p) = 0$$

Parametric calculation. Let $z(t) = e^{it} + 2$. Then $v(t) = ie^{it}$.

$$\oint_C (1/z)\, dz = \int_0^{2\pi} (e^{it} + 2)^{-1}\, ie^{it}\, dt = i \int_0^{2\pi} \frac{e^{it}}{(e^{it}+2)}\, dt = \log\left[2 + e^{i\,t}\right]_0^{2\pi} \text{ (by } \textit{Mathematica}\text{)}$$

$$= [\ln|2 + e^{i\,t}| + i\, \mathrm{Arg}(2 + e^{i\,t})]_0^{2\pi}$$

= ln 3 + i Arg(3) - (ln 3 - i Arg(3)) = 0

The parametric calculation confirms the result from (6).

We try integration by parts:

Let u = $(e^{it} + 2)^{-1}$, v = e^{it}

du = $-(e^{it} + 2)^{-2} i e^{it} dt$

dv = $i e^{it}$ dt

$i\int_0^{2\pi} u\,dv = i\left[\left((e^{it} + 2)^{-1} e^{it}\right)_0^{2\pi} - \int_0^{2\pi} -i\, e^{it} (e^{it} + 2)^{-2}\, dt\right]$

$= 0 + \int_0^{2\pi} e^{it} (e^{it} + 2)^{-2}\, dt = \ldots$

This is an infinite regress with ever larger negative powers n of $(e^{it} + 2)^{-n}$, which implies that the integral disappears.

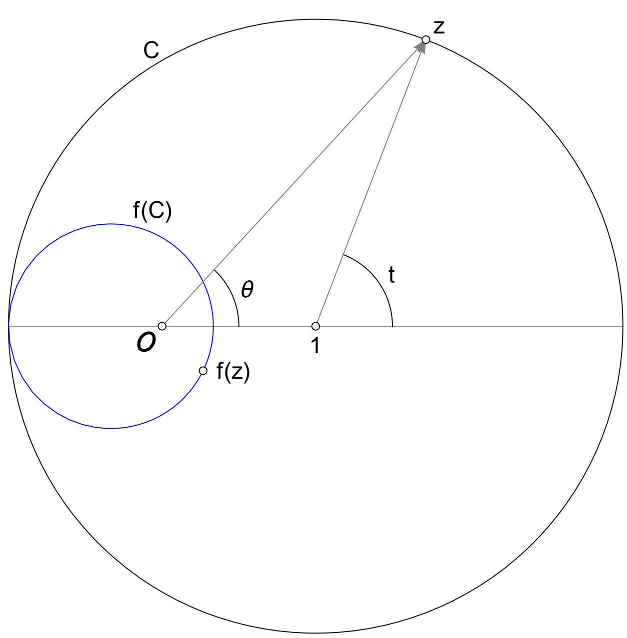

Figure 2: |z−1| = 2, f(z) = 1/z

(iii) |z-1| = 2

The winding number for z is v(C, 0) = 1. Figure 2 shows that z winds around the origin. Then from (7),

$\oint_C (1/z)\, dz = 2\,\pi\, i\; v(C, 0) = 2\,\pi\, i$

Parametric evaluation.

In (i) we calculated the contour integral of a circle that winds around the origin. This circle has twice the diameter of the circle in (i). Does that matter? Not according to the discussion of contour integrals at the top of p. 391. But this is a parametric evaluation.

Let $z(t) = 2e^{it} + 1$. Then $v(t) = 2\,ie^{it}$.

$\oint_C (1/z)\, dz = 2\,i \int_0^{2\pi} \frac{e^{it}}{2\,e^{it}+1}\, dt = \log\left[1 + 2\,e^{i\,t}\right] \big|_0^{2\pi}$ (*Mathematica*)

$= (\ln(3) + i\text{Arg}(1+2\,e^{i\,2\,\pi})) - (\ln(3) + i\text{Arg}(1+2\,e^{i\,0})) = (0 + i\ \text{Arg}(3) - i\ \arg(3)) = 0$

This direct parametric evaluation gives the wrong answer, because Figure 2 shows that 0 is inside C, so there should be a wind around C, leading to an answer of 2π i. Vasco has shown how to obtain the correct answer. It helps to consider that the integral is the sum of the signed angles subtended by Δ_i for all i. The horizontal (real) parts cancel (p. 391). Then we need only consider the angular components of log. At the two points 2π and 0, the angle $\theta(t)$ is equivalent to t. Vasco has:

$z = 1 + 2\,e^{it} = R(t)\,e^{i\theta(t)}$

$\log(1 + 2\,e^{it}) = \log(R(t)\,e^{i\theta(t)}) = \ln(R(t)) + i\theta(t)$

Vasco then reasoned that "the net increase in θ as to goes from 0 to 2π is $\theta(2\pi) - \theta(0) = 2\pi$". Then, $i[\theta(2\pi) - \theta(0)] = 2\pi$ i.

Using *Mathematica* to calculate $2\,i \int_0^{2\pi} \frac{e^{it}}{2\,e^{it}+1}\, dt$, the result is $2\,\pi$ i.

Mathematica calculations for (ii)

```
Clear["Global`*"]

z = Exp[I t] + 2;

dz = I Exp[I t] ;

Print[" ∮_C (1 / z) dz : ", Integrate[(1 / z) dz, t]]

Print[" ∮_C (1 / z) dz : ", Integrate[(1 / z) dz, {t, 0, 2 Pi}]]
```

$\oint_C (1/z)\, dz$: $\text{Log}\left[2 + e^{i\,t}\right]$

$\oint_C (1/z)\, dz$: 0

Mathematica calculations for (iii)

```
In[1]:= Clear["Global`*"]

z = 2 Exp[I t] + 1;

dz = 2 I Exp[I t] ;

Print[" ∮_c (1 / z) dz : ", Integrate[(1 / z) dz, t]]
Print[" ∮_c (1 / z) dz : ", Integrate[(1 / z) dz, {t, 0, 2 Pi}]]
```

$\oint_c (1 / z)\, dz$: $\mathrm{Log}\left[1 + 2\, e^{i\, t}\right]$

$\oint_c (1 / z)\, dz$: $2\, i\, \pi$