Needham, Chapter 8, Exercise 26, p. 426.
G. Palmer, August 30, 2017, 7:20 am PDT

26 (i) Show that if f is an analytic function without singularities or p-points on a loop L, then

$$v[f(L), p] = \frac{1}{2\pi i} \oint_L \frac{f'(z)}{f(z) - p}\, dz$$

We have to recognize that $\frac{f'(z)}{f(z)-p}$= [log $(f(z) - p)$]'. Then

$$\oint_L \frac{f'(z)}{f(z)-p}\, \mathrm{dz} = \oint_L [\log(f(z) - p)]'\, \mathrm{dz} = [\log \mathrm{f(z)} - \mathrm{p}]_{z_1}^{z_2}$$

where z_1 and z_2 are complex numbers representing the starting and finishing points of z on L.

$$[\log \mathrm{f(z)} - \mathrm{p}]_A^A = [\ln |\mathrm{f(z)} - \mathrm{p}| + i \arg(\mathrm{f(z)} - \mathrm{p})]_{z_1}^{z_2}$$

$$= [\ln |f(z_2) - p| + i \arg(f(z_2) - p)] - [\ln |f(z_1) - p| + i \arg(f(z_1) - p)]$$

$$= i\, [\arg(f(z_2) - p) - \arg(f(z_1) - p)] = i \cdot 2n\pi$$

When z traverses L from z_1 to z_2, f(z_2) falls on f(z_1), but f(L) rotates through an angle of n2π, where n is the number of revolutions of f on L round p. Then, by the definitions of winding numbers on pp. 338-339, n = v[f(L), p], and with minor rearranging, the equation is proven.

$$\oint_L \frac{f'(z)}{f(z)-p}\, \mathrm{dz} = 2n\pi\, \mathrm{i} = 2\pi\, \mathrm{i}\, \mathrm{v[f(L), p]}$$

$$v[f(L), p] = \frac{1}{2\pi \mathrm{i}} \oint_L \frac{f'(z)}{f(z)-p}\, \mathrm{dz}$$

Note that there is no need to parameterize the integrand. The integral can be found using the Fundamental Theorem. There is a temptation to write the interval in terms of an angle, say θ, but then the endpoints would have to be defined as angles, which are real numbers, and the integrand would have to be parameterized.

(ii) Now let

$$f(z) = \left((z - a_1)^{A_1} (z - a_2)^{A_2} \cdots (z - a_n)^{A_n}\right) \Big/ \left((z - b_1)^{B_1} (z - b_2)^{B_2} \cdots (z - b_m)^{B_m}\right)$$

and by considering (log f)', find (f'/f).

Answer:

Rewrite f(z) for convenience.

$$\mathrm{f(z)} = \prod_{i=1}^{n} (z - a_i)^{A_i} / \prod_{j=1}^{m} (z - b_i)^{B_i}$$

$$\log \mathrm{f} = \sum_{i=1}^{n} A_i \log(z - a_i) - \sum_{j=1}^{m} B_j \log(z - b_i)$$

$\frac{f'}{f}$ = (log f)' = $\sum_{i=1}^{n} \frac{A_i}{z-a_i}$ - $\sum_{j=1}^{m} \frac{B_j}{z-b_j}$

(iii) In part (i), put p = 0 and take L to be a simple loop containing the roots a_1 to a_r and containing the poles b_1 to b_s. Thereby obtain a calculation proof of the Generalized Argument Principle in the case of rational functions.

ν[f(L), 0] = $\sum_{j=1}^{r} A_j$ - $\sum_{j=1}^{s} B_j$

In part (i), put p = 0...

v[f(L), 0] = $\frac{1}{2\pi i} \oint_L \frac{f'(z)}{f(z)}$ dz

= $\frac{1}{2\pi i} \oint_L$ ($\sum_{j=1}^{r} \frac{A_j}{z-a_j}$ - $\sum_{j=1}^{s} \frac{B_j}{z-b_j}$) dz by substitution from (ii)

= $\frac{1}{2\pi i}$ ($\sum_{j=1}^{r} A_j \oint_L \frac{1}{z-a_j}$dz - $\sum_{j=1}^{s} B_j \oint_L \frac{1}{z-b_j}$ dz) $\Sigma \int f_i = \int \Sigma f_i$

The subscripts for the two terms have shifted from (i, j) to (j, j). It doesn't matter here, because the j's belong to different parameters (s, r) and vary independently on the sums. From (7) we know $\oint_L \frac{1}{z-p}$ dz = 2πi ν(L, p). Since this is a simple loop, we know that $\nu(L, a_j)$ = 1 for all j in r and $\nu(L, b_j)$ = 1 for all j in s. Then,

v[f(L), 0] = $\frac{1}{2\pi i}$ ($\sum_{j=1}^{r} A_j$ 2πi - $\sum_{j=1}^{s} B_j$ 2πi)

= $\sum_{j=1}^{r} A_j$ - $\sum_{j=1}^{s} B_j$