Needham, Chapter 8, Exercise 25, p. 425-426.
G. Palmer, August 24, 2017, 4:15 pm PST

25 Consider the image of the disc $|z| \le R$ under the mapping $z \mapsto k\,z^m$. As the radius sweeps round the disc once, its image sweeps m times round the image disc of radius $|k|\,R^m$. Thus we may sensibly define the area of the image to be $m\pi\,(|k|\,R^m)^2$. With this understanding, show that if a mapping has a convergent power series

$$f(z) = a + bz + cz^2 + dz^3 + \cdots,$$

then the area of the image is just the sum of the areas of the images under each of the separate terms of the series:

$$\text{area of image} = \pi\left(\,|\,b\,|^2 R^2 + 2\,|\,c\,|^2 R^4 + 3\,|\,d\,|^2 R^6 + \cdots\right).$$

This is Bieberbach's Area Theorem.

Hint: Recall that the local area expansion factor is $|\,f'\,|^2$, so the image area is

$$\iint_{|z|\le R} |\,f'\,|^2 dx\,dy = \int_0^R \left[\int_0^{2\pi} f'\left(re^{i\theta}\right)\overline{f'\left(re^{i\theta}\right)}\,d\theta\right] r\,dr.$$

Answer using the hint:

$$f'(z) = (a + bz + cz^2 + dz^3 + \cdots)'$$

$$= b + 2cz + 3\,dz^2 + \cdots$$

$$= b + 2\,cre^{i\theta} + 3\,dr^2\,e^{i2\theta} + \cdots$$

$$\overline{f'(z)} = \bar{b} + \overline{2\,cz} + \overline{3\,dz^2} + \cdots$$

$$= \bar{b} + \overline{2\,c\,re^{i\theta}} + \overline{3\,d\,r^2\,e^{i2\theta}} + \cdots$$

Then we can write the product by multiplying the conjugate series $\overline{f'(z)}$ by every term of f'(z):

$$f'(z)\overline{f'(z)} = b(\,\bar{b} + \overline{2\,cz} + \overline{3\,dz^2} + \cdots) + 2\,cz(\bar{b} + \overline{2\,cz} + \overline{3\,dz^2} + \cdots) + 3\,dz^2(\bar{b} + \overline{2\,cz} + \overline{3\,dz^2} + \cdots)$$

$$= b\bar{b} + b\overline{2\,cz} + b\,\overline{3\,dz^2} + \cdots + 2\,cz\bar{b} + 2\,cz\overline{2\,cz} + 2\,cz\overline{3\,dz^2} + \ldots + 3\,dz^2\bar{b} + 3\,dz^2\overline{2\,cz} + 3\,dz^2\,\overline{3\,dz^2}$$

$$= |\,b\,|^2 + 2b\overline{cz} + 3b\overline{dz^2} + \cdots + 2c\bar{b}z + 4\,|\,c\,|^2\,|\,z\,|^2 + 6\,cd\,z\,\overline{z^2} + \cdots + 3\,d\,\bar{b}\,z^2 + 6\,d\,\bar{c}\,z^2\bar{z} + 9\,|\,d\,|^2\,|\,z\,|^4 + \cdots$$

Now to get terms that can be used in the double integral, we write $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$, where r is a real number, so it doesn't require the conjugate sign. Any term containing a factor z^n or $\overline{z^n}$ can be written $\alpha r^n e^{in\theta}$ or $\alpha r^n e^{-in\theta}$. Their integrals go to zero on a closed loop:

$$\int_0^{2\pi} \alpha r^n e^{in\theta} d\theta = \alpha r^n [\frac{e^{in\theta}}{in}]_0^{2\pi} = \alpha r^n [\frac{1}{in} - \frac{1}{in}] = 0$$

$$\int_0^{2\pi} \alpha r^n e^{-in\theta} d\theta = \alpha r^n [\frac{e^{-in\theta}}{in}]_0^{2\pi} = \alpha r^n [\frac{1}{in} - \frac{1}{in}] = 0$$

[Should this be parameterized as $\int_0^{2\pi} \alpha r^n e^{in\theta} (ine^{in\theta}) d\theta$? It still comes out 0.]

Then we are left with the diagonal terms:

$$f'(z)\overline{f'(z)} = |b|^2 + 4 |c|^2 |z|^2 + 9 |d|^2 |z|^4$$

$$= |b|^2 + 4 |c|^2 r^2 + 9 |d|^2 r^4$$

Apply the interior integral:

$$\int_0^{2\pi} f'(re^{i\theta})\overline{f'(re^{i\theta})} d\theta = \int_0^{2\pi} b\overline{b} d\theta + \int_0^{2\pi} 4 |c|^2 r^2 d\theta + \int_0^{2\pi} 9 |d|^2 r^4 d\theta + \cdots$$

$$= [|b|^2 \theta + 4 |c|^2 r^2 \theta + 9 |d|^2 r^4 \theta + \cdots]_0^{2\pi}$$

$$= 2\pi [|b|^2 + 4 |c|^2 r^2 + 9 |d|^2 r^2]$$

Apply the exterior integral to the result.

$$2\pi \int_0^R [|b|^2 + 4 |c|^2 r^2 + 9 |d|^2 r^4] r\, dr$$

$$= 2\pi [|b|^2 \frac{r^2}{2} + 4 |c|^2 \frac{r^4}{4} + 9 |d|^2 \frac{r^6}{6}]_0^R$$

$$= \pi (|b|^2 R^2 + 2 |c|^2 R^4 + 3 |d|^2 R^6 + \cdots)$$

When π is multiplied through, each of the terms equals $m\pi (|k| R^m)^2$, showing that *the area of the image is just the sum of the areas of the images under each of the separate terms of the series*.

But there is an easier way to prove the theorem using the "Area Interpretation" (pp. 393-394).

$$\text{Area}(f(z)) = \frac{1}{2i} \int \overline{f(z)} dz = \frac{1}{2i} \int \left(\overline{a + bz + cz^2 + dz^3} + \cdots\right) dz$$

$$= \frac{1}{2i} [\int \overline{a} dz + \int \overline{bz} dz + \int \overline{cz^2} dz + \int \overline{dz^3} dz + \cdots]$$

$$= \text{Area}(a) + \text{Area}(bz) + \text{Area}(cz^2) + \text{Area}(dz^3) + \cdots$$

Substitute for $m\pi\,(|k|\,R^m)^2$ in each term

$$= 0 + 1\,\pi\,\,|\,b\,|^2\,R^2 + 2\,\pi\,\,|\,c\,|^2\,R^4 + 3\,\pi\,|\,d\,|^2\,R^6 + \cdots$$

$$= \pi\left(\,|\,b\,|^2\,R^2 + 2\,|\,c\,|^2\,R^4 + 3\,|\,d\,|^2\,R^6 + \cdots\right)$$