Needham, Chapter 8, Exercise 23, p. 425.
G. Palmer, August 24, 2017, 3:15 pm PST

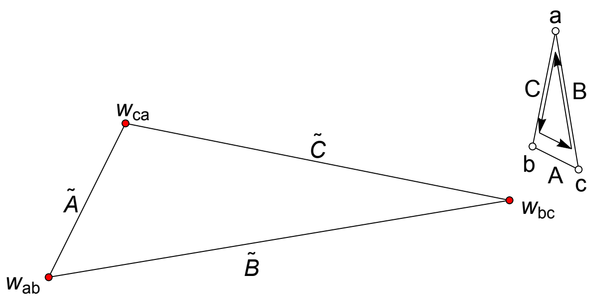

Figure 1. $w_{pq} = \dfrac{1}{(q-p)} \displaystyle\int_p^q f(z)\,dz$

23 Let f(z) be analytic throughout a region which contains a triangle with vertices a, b, c, and hence with edges A ≡ (c-b), b ≡ (a-c), C ≡ (b-a). Given a pair of point[s] p and q, let us define w_{pq} as a kind of average of f(z) along the line-segment pq:

$$w_{pq} = \frac{1}{(q-p)}\int_p^q f(z)\,dz.$$

Show that this complex average mapping sends the sides of the triangle abc to the vertices w_{ab}, w_{bc}, w_{ca} of a similar triangle! ...
[Hint: Show that $Aw_{bc} + Bw_{ca} + Cw_{ab} = 0$, and use $A + B + C = 0$.

Figure 1 illustrates the problem using f(z) = z^2, which is analytic throughout the C plane. It's easy to see that A + B + C = 0. We can also see that

$$Aw_{bc} + Bw_{ca} + Cw_{ab}$$

$$= \frac{c-b}{c-b}\int_b^c f(z)\,dz + \frac{a-c}{a-c}\int_c^a f(z)\,dz + \frac{b-a}{b-a}\int_a^b f(z)\,dz$$

$$= \int_b^c f(z)\,dz + \int_c^a f(z)\,dz + \int_a^b f(z)\,dz$$

$$= \oint_K f(z)dz = 0,$$

by Cauchy's theorem, because this is an integral on a closed loop in a region where f(z) is analytic. To show that the triangles are similar, we want to show that the ratio of the lengths of any two sides of triangle w_{ab}, w_{bc}, w_{ca} are equal to the ratios of corresponding sides of a, b, c, or that the ratios of two sides are equal and the angles between them are equal. Using A + B + C = 0, we can write A = -B - C, B = -A - C, C = -A - B.

$$(-B - C)\,w_{bc} + B\,w_{ca} + Cw_{ab} = 0$$

$$B(w_{ca} - w_{bc}) + C(w_{ab} - w_{bc}) = 0$$

$$B = -\frac{C(w_{ab}-w_{bc})}{(w_{ca}-w_{bc})} = \frac{C(w_{ab} - w_{bc})}{(w_{bc} - w_{ca})}$$

$$\frac{B}{C} = \frac{w_{ab} - w_{bc}}{w_{bc} - w_{ca}} = re^{i\theta},$$

The fact that the points on the left and centre of the equation are equal means that we can write them as $re^{i\theta}$, which shows immediately that the ratios of two sides (r) are equal and the angles between them are the same. The triangles are therefore similar.

I only stopped here when I read Vasco’s solution. Originally, I continued and showed that the ratios of two corresponding sides were the same for the sides at all three vertices of both triangles.

$$\Longrightarrow \frac{|B|}{|C|} = \frac{|w_{ab} - w_{bc}|}{|w_{bc} - w_{ca}|}, \text{ etc.}$$

That was a lot more work than writing $re^{i\theta}$!