Needham, Chapter 8, Exercise 22, p. 424.
G. Palmer, August 27, 2017, 3:20 PDT

22 Let K be the short contour of the previous exercise. Suppose that f(z) possesses a Taylor series centred at a that converges at points of K:

$$f(a+h) = f(a) + \frac{f'(a)}{1!} h + \frac{f''(a)}{2!} h^2 + \frac{f'''(a)}{3!} h^3 + \cdots.$$

[The existence of such a series for any analytic function is derived in the next chapter.]

(i) By integrating this series along K, show that the difference between the exact integral and the R_M value is roughly $\frac{1}{24}f''(a)\ \epsilon^3$. Verify that the results of the first two parts of the previous exericise are in accord with this finding.

The coefficients $\frac{f^{(n)}(a)}{n!}$ are constants. The variable h stands in for z in a more general complex equation.

Let FT = $\int_{-\epsilon/2}^{\epsilon/2} f(a+h)\, dh$

$$= \int_{-\epsilon/2}^{\epsilon/2} f(a)\, dh + \int_{-\epsilon/2}^{\epsilon/2} \frac{f'(a)}{1!} h\, dh + \int_{-\epsilon/2}^{\epsilon/2} \frac{f''(a)}{2!} h^2\, dh + \int_{-\epsilon/2}^{\epsilon/2} \frac{f'''(a)}{3!} h^3\, dh + \cdots$$

$$[f(a)\, h + \frac{f'(a)}{2} h^2 + \frac{f''(a)}{2!\ \ 3} h^3 + \cdots]_{-\epsilon/2}^{\epsilon/2}$$

$$= [f(a)\ (\epsilon/2\ - (-\epsilon/2))] + [\ \frac{f'(a)}{2}\ ((\epsilon/2)^2 - (-\epsilon/2)^2)\ ] + [\frac{f''(a)}{6}\left((\epsilon/2)^3 - (-\epsilon/2)^3\right)] + \cdots$$

$$= f(a)\ \epsilon\ + \frac{f''(a)}{6} \frac{\epsilon^3}{4} + \cdots$$

$$\simeq\ f(a)\ \epsilon\ +\ \frac{f''(a)}{24} \epsilon^3$$

Error = FT - R_M

$$=\ f(a)\ \epsilon\ +\ \frac{f''(a)}{24} \epsilon^3\ -\ f(a)\ \epsilon$$

$$= \frac{f''(a)}{24} \epsilon^3$$

Verify that the results of the first two parts of the previous exericise are in accord with this finding.

Result of Ex. 21 (i): The error induced by R_M in the integral of z^2 along K is $\frac{\epsilon^3}{12}$.

$\frac{f''(a)}{24} \epsilon^3 = \frac{(z^2)''(a)}{24} \epsilon^3 = \frac{2}{24}\epsilon^3 = \frac{\epsilon^3}{12}$ The result of the first part is verified.

Result of Ex. 21 (ii): The error induced by R_M in the integral of e^z along K is $\frac{1}{24} e^a \epsilon^3$.

$\frac{f''(a)}{24} \epsilon^3 = \frac{e^a}{24} \epsilon^3$ The result of the second part is verified.

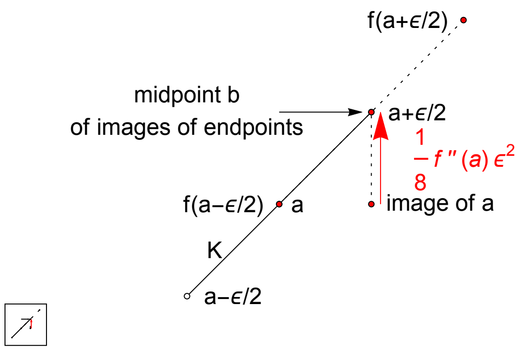

Figure 1. Distance from f(a) to midpoint b
of images of ends of K, f(z) = e^z

(ii) Use the series to show that the complex number from the image of the midpoint of K to the midpoint of the images of the ends of K is roughly $\frac{1}{4}$f''(a) ϵ^2. As ϵ shrinks, are these two types of midpoint distinguishable under the magnifying lens that produces figure [28]?

Figure 1 illustrates the construction of points and images using e^z for f. The image of the midpoint of K is f(a). The image of any other point of K is given by

$$f(a+h) = f(a) + \frac{f'(a)}{1!} h + \frac{f''(a)}{2!} h^2 + \frac{f'''(a)}{3!} h^3 + \cdots.$$

In particular, the endpoints are given by the following two equations. To find the midpoint b, we add them and divide by 2 to get the image midpoint.

$$f(a+\epsilon/2) = f(a) + \frac{f'(a)}{1!} (\epsilon/2) + \frac{f''(a)}{2!} (\epsilon/2)^2 + \frac{f'''(a)}{3!} (\epsilon/2)^3 + \cdots.$$

$$f(a-\epsilon/2) = f(a) + \frac{f'(a)}{1!} (-\epsilon/2) + \frac{f''(a)}{2!} (-\epsilon/2)^2 + \frac{f'''(a)}{3!} (-\epsilon/2)^3 + \cdots.$$

b = (1/2) [2f(a) + $2\frac{f''(a)}{2!} (\epsilon/2)^2$] (only terms with h^n, n even, survive)

= [f(a) + $\frac{f''(a)}{2!} (\epsilon/2)^2$]

$= f(a) + \frac{f''(a)}{8}\epsilon^2$

The difference we seek is b - f(a) = (f(a) + $\frac{f''(a)}{8}\epsilon^2$) - f(a) = $\frac{1}{8}$f''(a) ϵ^2. As Vasco pointed out, the answer in the text of $\frac{1}{8}$f''(a) ϵ^2 is an error. This number is plotted for e^z as the red arrow in Figure 1, showing that it equals the difference between the plotted points f(a) and the midpoint b.

As ϵ shrinks, are these two types of midpoint distinguishable under the magnifying lens that produces figure [28]?

We assume that the magnifying glas that produces figure [28] is L_1, “which is so powerful that we can actually see [a disc of radius ϵ] with it”. As ϵ shrinks, the complex number connecting f(a) to the midpoint of the endpoints shrinks as ϵ^2, so they would soon become indistinguishable in an ϵ^2-shrunken version of [28]. Shrinking ϵ to .1ϵ would make the image an order of 1/100 the area. This order of shrinkage is illustrated by the inset on the lower left in Figure 1. Since the magnifying glass in [28] appears to be about half of the one in the unshrunken part of Figure 1, it would probably be indistinguishable, or nearly so. It would be 1/4 the area of the inset in Figure 1. Shrinking to .01ϵ would make it indistinguishable. We would require a glass of power L_3 to view it.

In Figure 1, we see that (f(a - ϵ/2) = a) and (f(a+ϵ/2) = b). These coincidences are probably artifacts of the construction of K and the choice of function.

(iii) From the Fundamental Theorem, deduce that the existence of such a series implies the vanishing of the integral of f round loops within the disc of convergence.

The phrase “such a series” must refer to

$$f(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \cdots.$$

This series applies to short segments on the vector h, where a is a starting point or a midpoint. In (i), we found the exact value of the integral of the series where a was the midpoint of a small complex number ϵ. The present problem poses using the series representation of an analytic function f(z) to deduce the vanishing of f(z) on a loop (which we will refer to as “C”, as in [27]) placed within the disc of convergence. We have to ensure that all arguments to f (all z points) fall within that disc so that f will converge to a finite value. Since we don’t know the exact boundaries of the disc of convergence, we need a test. A crude but obvious test is provided by the fact that all values falling on or within C are by definition within the disc of convergence. Since we have to use the series *f(a+h)*, we have to construct a situation on or within C analogous to that in (i), in which there are small numbers representing straight segments. Perhaps we could integrate these segments to get an approximation of C. That suggests that the segments might be either tangents or chords of C. They can’t be tangents, because their endpoints might fall outside the disc of convergence. Not knowing where the boundaries lie without knowing f, we would have no control over that. So we plot chords ϵ_j of C end to end round C. These chords have the convenient feature that their endpoints fall on C and their midpoints (a_j) fall in the interior of C, so the endpoints and midpoints fall within the disc of convergence, and in aggregate the chords form a closed loop K. The construction now fits the description of [27], p. 411, in which K is the boundary of the shaded region. By letting the chords k_j shrink, K approximates C (p. 411, top, and Ex. 20 (iv)).

To obtain an integral for K, we use f(a+h) as defined above and use the Fundamental Theorem to integrate f on every k_j and then sum the integrals. For each k_j, we integrate on the intervals $\left(a_j - \epsilon_j/2\right)$ to $\left(a_j + \epsilon_j/2\right)$.

$$\oint_K \mathrm{f(z)\ dz} = \sum_{j=1}^{n} \int_{a_j-\epsilon_j/2}^{a_j+\epsilon_j/2} \mathrm{f(z)\ dz}$$

We know by inspection that f(a+h) is an analytic function which is the sum of a series of analytic power functions in h, none of which contain 1/h. We reason from the discussion of power functions on pp. 396-398 that integrals of power functions with powers other than (-1) must vanish on closed loops. By applying the Fundamental Theorem to every chord, the integral of any term of f(a+h) must vanish on a closed loop, even if the closed loop contains the origin, and even if it is not a simple, smooth loop.† Every term of the integral of $f(a_i + h_i)$ vanishes on K. Therefore, the integral of f(a+h) itself must vanish on K.

By the Deformation Theorem (11), p. 398: *If a contour sweeps only through analytic points as it is deformed, the value of the integral does not change*. We can regard C as a deformation of K because both lie within the disc of convergence where f(a+h) is analytic and finite. Hence, if the integral of f on K vanishes, then so must the integral of f on C.

We have shown that the integral of every summand must vanish on K. In fact, just knowing that f(a+h) is analytic and contains no residue summand with a term 1/h is sufficient to show that f(a+h) vanishes on K. Then we can either apply the Deformation Theorem, or, as required by the problem statement, we can permit ϵ to shrink and apply the Fundamental Theorem.

Then, for k_j, j = 1..n, as $\epsilon_j \longrightarrow 0$,

$$\oint_C \mathrm{f(z)\ dz} = \oint_K \mathrm{f(z)\ dz} = \sum_{j=1}^{n} \int_{a_j-\epsilon_j/2}^{a_j+\epsilon_j/2} \mathrm{f(z)\ dz} = 0,$$

showing that the existence of f(a+h) implies the vanishing of the integral of f round loops within the disc of convergence.

†Though K has many more chords than a square has sides, the situation is analogous to the vanishing of an analytic function round the square in [28], p. 412.