Needham, Chapter 8, Exercise 21, p. 424.
G. Palmer, August 10, 2017, 12:30 PST

21 Let K be the straight contour from a-(ϵ/2) to a + (ϵ/2), where ϵ is a short complex number in an arbitrary direction.

(1) Use the Fundamental Theorem to integrate z^2 along K, and then write down the value obtained by using a single term in R_m. Show that the error induced by R_M is $\frac{1}{12}\epsilon^3$.

FT (Fundamental Theorem)

Let B = a+(ϵ/2) and A = a-(ϵ/2)

$$\int_A^B z^2\, dz = [\tfrac{z^3}{3}]_A^B$$

$$= \frac{(a+(\epsilon/2))^3}{3} - \frac{(a-(\epsilon/2))^3}{3}$$

$$= \left(a^3 + \frac{3\epsilon a^2}{2} + \frac{3\epsilon^2 a}{4} + \frac{\epsilon^3}{8}\right)/3 - (a^3 - \frac{3\epsilon a^2}{2} + \frac{3\epsilon^2 a}{4} - \frac{\epsilon^3}{8})/3$$

$$= a^2\epsilon + \frac{\epsilon^3}{12}$$

R_M (z is midpoint of Δ)

$$R_M = w\,\Delta = z^2\,\epsilon = a^2\,\epsilon$$

$$\text{Error} = (\text{FT} - R_M) = \left(a^2\epsilon + \frac{\epsilon^3}{12}\right) - a^2\epsilon = \frac{\epsilon^3}{12}$$

(ii) As in part (i), find both the exact value and the R_M value for the integral of e^z along K. By expanding $e^{\epsilon/2}$ as a power series, deduce that the error in this case is roughly $\frac{1}{24}\epsilon^a\epsilon^3$

FT

$$\int_{a-(\epsilon/2)}^{a+(\epsilon/2)} e^z\, dz = [e^z]_{a-(\epsilon/2)}^{a+(\epsilon/2)}$$

$$= e^{a+(\epsilon/2)} - e^{a-(\epsilon/2)} = e^a\left(e^{\epsilon/2} - e^{-\epsilon/2}\right)$$

$$= e^a[1 + \epsilon/2 + \frac{1}{2!}(\epsilon/2)^2 + \frac{1}{3!}(\epsilon/2)^3 + \frac{1}{4!}(\epsilon/2)^4 + \cdots$$

$$- (1 + (-\epsilon/2) + \frac{1}{2!}(-\epsilon/2)^2 + \frac{1}{3!}(-\epsilon/2)^3 + \frac{1}{4!}(-\epsilon/2)^4 + \cdots)]$$

$$= e^a(\epsilon + \frac{2}{3!}(\epsilon/2)^3 + \cdots) \simeq e^a\epsilon + \frac{1}{24}e^a\,\epsilon^3$$

R_M (z is midpoint, a)

$R_M = \text{w } \Delta = e^z\, \epsilon = e^a\, \epsilon$

Error = (FT - R_M) = $(e^a \epsilon + \frac{1}{24} e^a\, \epsilon^3) - e^z\, \epsilon$

$= \frac{1}{24} e^a\, \epsilon^3$

(iii) Repeat the error analysis of the previous parts for the non-analytic function $\overline{z}^2$. [You will need to use parametric evaluation to find the exact value of the integral.]

PE (parametric evaluation)

$z(t) = (a - \epsilon/2) + \epsilon\, t$

$dz(t) = \epsilon$

$f(t) = \overline{z}^2 = ((\overline{a} - \overline{\epsilon}/2) + \overline{\epsilon}\, t)^2$

$f(t) dz(t) = ((\overline{a})^2 + 2\, t\, \overline{\epsilon}\, \overline{a} - \overline{\epsilon}\, \overline{a} + t^2\, (\overline{\epsilon})^2 - t\, (\overline{\epsilon})^2 + \frac{(\overline{\epsilon})^2}{4}) \epsilon$

$= (\overline{a})^2 \epsilon + 2\, \overline{a}\, |\, \epsilon\, |^2\, t - |\, \epsilon\, |^2\, \overline{a} + |\, \epsilon\, |^2\, \overline{\epsilon}\, t^2 - t\, |\, \epsilon\, |^2\, \overline{\epsilon} + \frac{|\epsilon|^2 \overline{\epsilon}}{4}$

$= (\overline{a})^2 \epsilon - |\, \epsilon\, |^2\, \overline{a} + \frac{|\epsilon|^2 \overline{\epsilon}}{4} + \left(2\, \overline{a}\, |\, \epsilon\, |^2 - |\, \epsilon\, |^2\, \overline{\epsilon}\right) t + |\, \epsilon\, |^2\, \overline{\epsilon}\, t^2$

Find $\int_0^1 f(z)\, dz$

$\left((\overline{a})^2\, \epsilon - |\, \epsilon\, |^2\, \overline{a} + \frac{|\epsilon|^2 \overline{\epsilon}}{4}\right) \int_0^1 dt + |\, \epsilon\, |^2\, (2\, \overline{a} - \overline{\epsilon}) \int_0^1 t\, dt + |\, \epsilon\, |^2\, \overline{\epsilon} \int_0^1 t^2\, dt$

$= [\left((\overline{a})^2\, \epsilon - |\, \epsilon\, |^2\, \overline{a} + \frac{|\epsilon|^2 \overline{\epsilon}}{4}\right) t + |\, \epsilon\, |^2\, (2\, \overline{a} - \overline{\epsilon})\, \frac{t^2}{2} + |\, \epsilon\, |^2\, \overline{\epsilon}\, \frac{t^2}{3}]_0^1$

$= \left((\overline{a})^2\, \epsilon - |\, \epsilon\, |^2\, \overline{a} + \frac{|\epsilon|^2 \overline{\epsilon}}{4}\right) + \frac{1}{2}\, |\, \epsilon\, |^2\, (2\, \overline{a} - \overline{\epsilon}) + \frac{1}{3}\, |\, \epsilon\, |^2\, \overline{\epsilon}$

$= (\overline{a})^2\, \epsilon - (|\, \epsilon\, |^2\, a - |\, \epsilon\, |^2\, a) + \left(\frac{|\epsilon|^2 \overline{\epsilon}}{4} - \frac{1}{2}\, |\, \epsilon\, |^2\, \overline{\epsilon} + \frac{1}{3}\, |\, \epsilon\, |^2\, \overline{\epsilon}\right)$

$= (\overline{a})^2\, \epsilon + (1/12)\, |\, \epsilon\, |^2\, \overline{\epsilon} = \epsilon\, (\overline{a}^2 + \frac{\overline{\epsilon}^2}{12})$

R_m

$R_M = \text{w}\,\Delta = \overline{z}^2\,\epsilon = \overline{a}^2\,\epsilon$

$\text{Error} = (\text{PE} - R_m) = ((\overline{a})^2\,\epsilon + \frac{1}{12}\,|\,\in\,|^2\,\overline{\in}) - \overline{a}^2\,\epsilon$

$= \frac{1}{12}\,|\,\in\,|^2\,\overline{\in}$