Needham, Chapter 8, Exercise 20, p. 424.
G. Palmer, August 8, 2017, 3:50 PST

20 In [27], consider the white fragments of squares sandwiched between K and C.

(i) Show that the sum of the integrals round these fragments equals the difference between the integrals round C and K.

Note: By "sum of the integrals" we took this to mean integration on any f(z), i.e. a general mapping (p. 411, line -4). This may not be the case, but the general reasoning is the same if we interpret it as f(z) = z.

The contour integral $\oint_C f(z)\,dz$ round C for all C not containing the origin is 0 for an analytic function that is not the complex inverse 1/z. Otherwise, its value may be found by The Fundamental Theorem of Calculus, by parametric evaluation, or in the case of f(z) = 1/z, by (6) or (7) (p. 392) in combination with residues, or in the case of $\overline{z}$, by (8).

The integral round K is the sum of the integrals for all the shaded squares of [27], given in (18). Due to cancellations of common sides of adjacent squares, only squares with non-touching sides contribute to the integral on K.

$$\oint_K f(z)\,dz = \sum_{\text{shaded squares}} \oint_\square f(z)\,dz$$

The fragments between C and K consist entirely of two to four-sided shapes. One of the sides of each fragment lies in C, which consists entirely of the sum of all such sides. f(z) traverses these sides in the same direction as C, the counterclockwise direction. Let c_i be a side of a fragment in C. We can write

$$\oint_C f(z)\,dz = \sum_{i=1}^{n} \oint_{c_i} f(z)\,dz \qquad \text{(sum integrals on all fragment contours in C)}$$

Of course, $\oint_C f(z)\,dz$ can also be found in other ways, as explained above.

Other sides of each fragment are shared with K, which is composed of the sum of all such sides shared with fragments. The direction of traversal on fragment sides shared with K is opposite to that of traversals on shaded squares in K. Therefore, for any fragment, the sum of these negative sides-in-K is given by ($-\oint_{\square_i} f(z)\,dz$), where the "i" indexes the fragment. The sum of all such sides is equivalent to ($-\sum_{\text{shaded squares}} \oint_\square f(z)\,dz$), which is ($-\oint_K f(z)\,dz$).

Some fragments have one or two sides that are neither in C nor shared with K. In Figure [27], these sides have white space on either side. When integrals are calculated for adjacent fragments, these sides are traversed in opposite directions. Sides between adjacent fragments cancel out.

The sum of the integrals round all fragments is given by

$$\oint_{\text{frag}} f(z)\,dz = \sum_{i=1}^{n} \left(\oint_{c_i} f(z)\,dz - \oint_{\square_i} f(z)\,dz\right) \qquad (1)$$

$$= \oint_C f(z)\,dz - \sum_{\text{shaded squares}} \oint_\square f(z)\,dz$$

$$= \oint_C f(z)\,dz - \oint_K f(z)\,dz,$$

which is the difference between the integrals round C and K.

Then

$$\oint_{\text{frag}} f(z)\,dz = \oint_C f(z)\,dz - \oint_K f(z)\,dz \qquad (2)$$

Vasco's solution is more elegant, as it involves treating $\oint_C f(z)\,dz$ as the sum of the integrals round all the shapes: the white fragments plus the shaded squares of K. Then all shared sides cancel and equation (2) falls out from subtracting $\oint_K f(z)\,dz$ from both sides and rearranging. The solution derived from (1) simply reveals another facet of the structure depicted in [28].

(ii) As ϵ shrinks, what is the approximate size of each term in the above series?

If "the above series" refers to $(\oint_C f(z)\,dz - \oint_K f(z)\,dz)$ then it is really the sum of two series with different numbers of terms. The approximate size of a term of either series depends in part on f(z) and in part on the shapes of the fragments or squares, but the order of magnitude is the same for both. On pp. 412, it is asserted that the magnitude of each term dies away as ϵ^2. This is explained in "2 The Explanation" on pp. 412-414. Let the sides of the extended full squares in C and the shaded squares in K to be of length ϵ. Under shrinkage, a side acquires length $|\,p\,|\,\epsilon$ for general mappings, where p is (for analytic mappings) a sum of images. Since p is proportional to ϵ, a term, which is a sum of two such vectors, has the approximate size ϵ^2. The order holds for general mappings. For example, if f(z) = $\overline{z}$, the integral round one square is $2\,i\epsilon^2$, so we could say that the size or area is ϵ^2.

(iii) Roughly, how many terms are there in the series.

The approach used here is given on p. 411, last paragraph and p. 412: "The number of terms grows as (*fixed area inside C, divided by the area of each square*), that is as $1/\epsilon^2$." This appears to apply to a general mapping (p. 412, ¶ 1, s 1). The area in the fragment series is the imaginary part of the sum of the integrals of z conjugate on the contours of all the fragments (p. 393, 2 Area Interpretation). The number of terms n is the total fixed area divided by the size of each term: ϵ^2.

$$n = \frac{\mathrm{Im}\left[\sum_{i=1}^{n}\left(\oint_{C_i} \overline{z}\,dz - \oint_{\square_i} \overline{z}\,dz\right)\right]}{2\,\epsilon^2} = \frac{\mathrm{Im}\left[\oint_C \overline{z}\,dz - \oint_K \overline{z}\,dz\right]}{2\,\epsilon^2} \quad [\overline{z}]$$

The area of the fragments with shrinking ϵ is approximated by $(L/\epsilon)\,\epsilon^2$, where L is the length of C. Hence, n depends on L/ϵ, as in Vasco's answer, which he derives immediately from inspection of the boundaries of white fragments on C. I realize that I have derived the number of fragments in the same way, but at least we can see that n derived by inspection is consistent with n derived from area.

Our equation of difference of contour integrals with the sum of integrals on fragments supports the correctness of this equation, because the order of the terms in numerator equals the order in the denominator, i.e. ϵ^2. The first quotient then obviously becomes approximately n.

One might argue that "The number of terms grows as (*fixed area inside C, divided by the area of each square*), that is as $1/\epsilon^2$" was intended to apply only to the integral on z-conjugate, but I don't think that is the case. The total number of terms in the integral of any function is determined by the number of squares integrated, which is determined by the area of the contour.

(iv) From the previous parts, what do you conclude about the difference between the integrals round C and K, as ϵ shrinks to nothing.

From (ii), as ϵ shrinks to nothing, the difference between terms in C and K also shrinks to nothing. Both integrals are proportional to ϵ^2, so the difference in the size of terms is proportional to ϵ^2 (or perhaps to ϵ^3; see p. 414, ¶ 3). However, the difference in the integrals also depends on the number of terms, which depends on $1/\epsilon^2$. Furthermore, the number of terms in $\oint_C f(z)\,dz$ is generally different from the number in $\oint_K f(z)\,dz$. We have shown in (ii) that the size of fragments under shinking ϵ depends on ϵ^2 and in (iii) that the number of fragments is approximate L/ϵ. Then,

$$\oint_C f(z)\,dz - \oint_K f(z)\,dz = \sum_1^n \oint_{\text{frag}_i} f(z)\,dz \simeq (\mathrm{L}/\epsilon)\,\epsilon^2 = \mathrm{L}\epsilon$$

As $\epsilon \longrightarrow 0$, $\mathrm{L}\epsilon \longrightarrow 0$.

Vasco's solution is more conventional and rigorous. Our approximation is a bit looser.