Needham, Chapter 8, Exercise 2, p. 420.
G. Palmer, June 6, 2017, 10:30 pm, PST

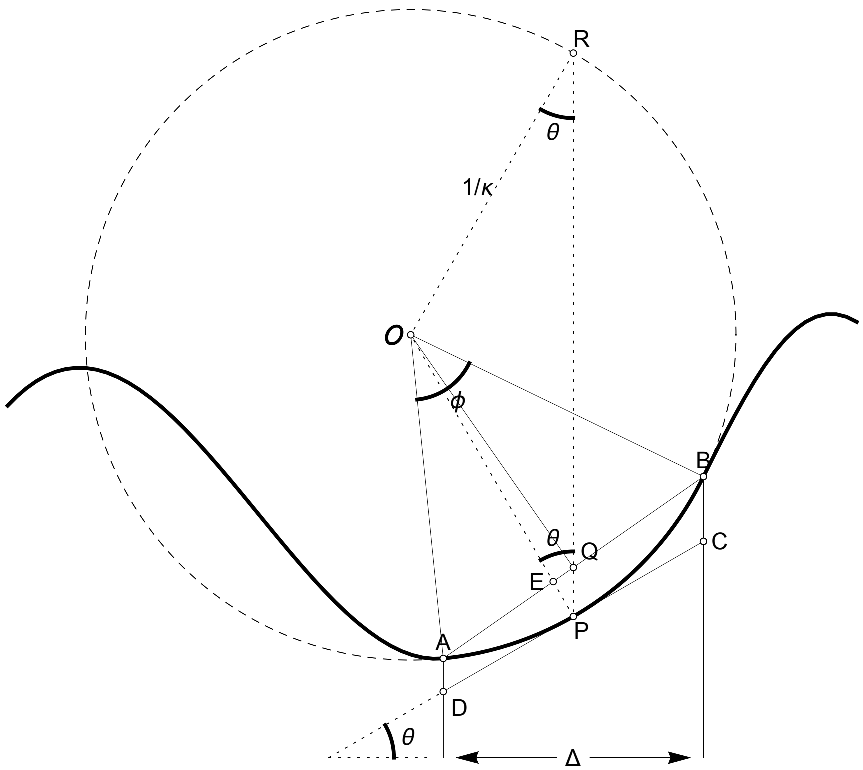

Figure 1: From [6], p. 383

2 In [6] show that

$$\lim_{\Delta \longrightarrow 0}\left(\frac{\text{area between the chord AB and the curve}}{\text{area between the tangent CD and the curve}}\right) = 2$$

In other words, R_M is twice as accurate as the Trapezoidal formula.

If one can calculate the area between the chord and the curve, one can subtract that from area(ABCD) to obtain the area between the tangent and the curve. The area of the pie-shaped slice under the curve segment AB is just $\pi r^2 \frac{\phi}{2\pi} = \frac{r^2\phi}{2} = \frac{\phi}{2k^2}$. We need the area of the triangle AOB, because areaUnder-Chord = areaPie - area(AOB).

As $\Delta \longrightarrow 0$, [AB] $\longrightarrow$ 2 DP.

DP = (1/2) Δ secθ (p. 382)

PR = $(2/\kappa)\cos\theta$ “

ϕ = [AB]/(1/k) = 2 DP/(1/k) = $\kappa\,\Delta\sec\theta$

area(Pie) = $\frac{\phi}{2k^2} = \frac{\kappa\,\Delta\sec\theta}{2k^2} = \frac{\Delta\sec\theta}{2k}$

The area of triangle A*O*B is (1/2) bh = (1/2) AB $\left(r^2 - \mathrm{DP}^2\right)^{1/2}$ = DP $\left(r^2 - \mathrm{DP}^2\right)^{1/2}$ as $\Delta \longrightarrow 0$.

area(A*O*B) = $(1/2)\,\Delta\sec\theta\,\left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}$

areaUnderChord = area(Pie) - area(A*O*B)

$$= \frac{\Delta\sec\theta}{2k} - (1/2)\,\Delta\sec\theta\,\left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}$$

areaUnderTan = area(ABCD) - areaUnderChord

$$= \left(\frac{1}{8}\kappa\sec^3\theta\right)\Delta^3 - \left[\frac{\Delta\sec\theta}{2k} - (1/2)\,\Delta\sec\theta\,\left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}\right]$$

$$\frac{\text{areaUnderChord}}{\text{areaUnderTan}} = \left(\frac{\Delta\sec\theta}{2k} - (1/2)\,\Delta\sec\theta\left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}\right) \Big/ \left(\frac{1}{8}\kappa\sec^3\theta\right)\Delta^3 - \left[\frac{\Delta\sec\theta}{2k} - (1/2)\,\Delta\sec\theta\left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}\right]$$

$$= \frac{\frac{1}{k} - \left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}}{\left(\frac{1}{4}\kappa\sec^2\theta\right)\Delta^2 - \left[\frac{1}{k} - \left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}\right]}, \text{ dividing by } (1/2)\,\Delta\sec\theta$$

$$= \frac{\frac{1}{k} - \left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}}{\left(\frac{1}{4}\kappa\sec^2\theta\right)\Delta^2 - \left[\frac{1}{k} - \left(1/k^2 - (1/4)\,\Delta^2\sec^2\theta\right)^{1/2}\right]}$$

Let $\Delta \longrightarrow 0$

$$= \frac{\frac{1}{k} - \left(1/k^2\right)^{1/2}}{-\left[\frac{1}{k} - \left(1/k^2\right)^{1/2}\right]} = \frac{\frac{1}{k} \mp \left(\frac{1}{k}\right)}{-\left[\frac{1}{k} \mp \left(\frac{1}{k}\right)\right]} = \{0, \text{indeterminant}, -\frac{\frac{2}{k}}{\frac{2}{k}}\} = \{0, \text{indeterminate}, -1\}$$