Needham, Chapter 8, Exercise 19, p. 424.
G. Palmer, August 5, 2017, 9 am PST

19 Let us verify the claim of Ex. 19, p. 262, that if a function has vanishing Schwarzian derivative, then it must be a Möbius transformation. Following Beardon[1984, p. 77], suppose that {f(z), z} = 0, and define F ≡ (f''/f').

What follows is my own working through the problems after consulting Vasco's solutions.

(i) Show that 1/F(z) - 1/F(w) = -(z-w)/2.

Since *F ≡ (f''/f')*, we can rewrite the Schwarzian derivative in terms of F.

$$F' - \frac{1}{2}F^2 = 0$$

$$\frac{dF}{dz} = \frac{1}{2}F^2$$

Anticipating that the RHS -(z-w)/2 will be created by integration of the form $c\int_z^w dz$, we rearrange with that in mind. Notice also that the summands on the LHS (*1/F(z) - 1/F(w)*) each have the format $\int \frac{1}{F^2}\,dF$. Hence, we could anticipate and write a target template: $\int_z^w \frac{1}{F^2}\,dF = c\int_z^w dz$.

$$\frac{dF}{F^2} = \frac{1}{2}dz$$

$$\int_z^w \frac{1}{F^2}\,dF = \frac{1}{2}\int_z^w dz$$

$$\left[-\frac{1}{F}\right]_z^w = \frac{1}{2}[z]_z^w$$

$$\left[-\frac{1}{F(w)} - \left(-\frac{1}{F(z)}\right)\right] = \frac{1}{2}(w-z)$$

$$\frac{1}{F(z)} - \frac{1}{F(w)} = -\frac{(z-w)}{2}$$

(ii) Deduce that $\frac{d}{dz}$ log f'(z) = -2/(z-a), for some constant a.

Noticing f' in the argument of log, we rewrite the derivative operator in terms of $\frac{d}{df'}$. This has two useful results: First, the derivative of log f' is $\frac{1}{f'}$; second, the residue is f''. The LHS $\frac{d}{dz}$ *log f'(z)* is now revealed to be $\frac{f''(z)}{f'(z)}$ in disguise. This seems like a good equation to remember for work with Schwarzian derivatives.

$$\frac{d}{dz} = \frac{df'}{dz}\cdot\frac{d}{df'} = f''\cdot\frac{d}{df'}$$

$$\frac{d}{dz}\log f'(z) = f''\cdot\left(\frac{d}{df'}\log f'(z)\right) = f''\cdot\frac{1}{f'} = \frac{f''(z)}{f'(z)} = F(z) \qquad (1)$$

$$\frac{1}{F(z)} = -\frac{(z-w)}{2} + F(w) = -\frac{z}{2} + \frac{w}{2} + F(w)$$

$$F(z) = \frac{1}{-\frac{z}{2}+\frac{w}{2}+F(w)} = \frac{2}{-z+w+2\,F(w)}$$

Let w have a constant value, which gives F(w) a constant value, and let a ≡ (w + 2F(w)). Then,

$$F(z) = -\frac{2}{(z-a)} \qquad (2)$$

From (1) and (2), we deduce that $\frac{d}{dz}$ *log f'(z) = -2/(z-a), for some constant a.*

(iii) Perform two further integrations to conclude that f(z) is a Möbius transformation.

Integrate the result of (ii), where c is a constant of integration.

$$\int[\frac{d}{dz}(-2\log(a-z))]dz = -2\int\frac{1}{z-a}dz = -2\log(z-a) + \log c = \log\frac{1}{(z-a)^2} + \log c = \log\frac{c}{(z-a)^2},$$

Then we can deduce that $f'(z) = \frac{c}{(z-a)^2}$. Integrate both sides.

$$f(z) = \int f'(z)\,dz = c\int\frac{1}{(z-a)^2}\,dz = \frac{-c}{z-a} + d \qquad (3)$$

Rewrite in Möbius format $\frac{Az+B}{Cz+D}$.

$$f(z) = \frac{-c+d(z-a)}{z-a} = \frac{dz-(ad+c)}{z-a} \qquad (4)$$

In Chapter 5, Section V, subsection 2 we find that a Möbius transformation can be specified by three points (complex numbers). In (4), A and D are specified by d and (-a), whose values must be chosen in some manner, either arbitrarily or because they are needed for some mathematical or practical purpose. C = 1. B = -(ad+c) = + [(-a) d - c], so B is fully specified by -a, d, and -c, up to and including the sign of B. Only three points are needed to specify f(z), so f(z) is a Möbius transformation. I don’t know whether it is always the case that the signs of the three points resolve in this way.