Needham, Chapter 8, Exercise 18, p. 423-424.
G. Palmer, July 26, 2017, 7:15 am PST

18 Let E be the elliptical orbit z(t) = a cos t + ib sin t, where a and b are positive and t varies from 0 to 2π. By considering the integral of (1/z) round E, show that

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{2\pi}{ab}$$

The 2π in the numerator of the RHS suggests that (6) has factored into the answer. This appears to be another exercise in which we parameterize, set the result equal to the contour integral, cancel outside (common) coefficients, take just the imaginary parts on both sides, and divide through by ab. The template is:

$$\int_0^{2\pi} \frac{v(t)}{z(t)} dt = \oint_E \frac{1}{z}\, dz = 2\pi \,i\, v(E, 0) = 2\pi\, i$$

Then

$$z(t) = a\cos t + ib\sin t$$

$$v(t) = -a\sin t + ib\cos t$$

$$\int_0^{2\pi} \frac{v(t)}{z(t)} dt = \int_0^{2\pi} \frac{-a\sin t + ib\cos t}{a\cos t + ib\sin t} dt = 2\pi\, i = \oint_E \frac{1}{z}\, dz$$

$$= \int_0^{2\pi} \frac{-a\sin t + ib\cos t}{a\cos t + ib\sin t}\left(\frac{a\cos t - ib\sin t}{a\cos t - ib\sin t}\right) dt$$

$$= \int_0^{2\pi} \frac{-a^2\cos t\sin t + abi\sin^2 t + abi\cos^2 t + b^2\cos t\sin t}{a^2\cos^2 t + b^2\sin^2 t} dt$$

$$= \int_0^{2\pi} \frac{(b^2-a^2)\cos t\sin t + abi}{a^2\cos^2 t + b^2\sin^2 t} dt = \int_0^{2\pi} \frac{(b^2-a^2)\cos t\sin t}{a^2\cos^2 t + b^2\sin^2 t} dt + \int_0^{2\pi} \frac{abi}{a^2\cos^2 t + b^2\sin^2 t} dt$$

This result must equal the contour integral. Equate real and imaginary parts. Since the contour integral has no real part, the first integral on the RHS equals 0, leaving

$$i\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t} dt = 2\pi i$$

$$\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2\sin^2 t} dt = \frac{2\pi}{ab} \qquad \text{(divide through by abi)}$$

This is what we wanted.