Needham, Chapter 8, Exercise 17, p. 423.
G. Palmer, July 25, 2017, 2 pm PST

17 Let

$$f(z) = \frac{1}{z}\left(z + \frac{1}{z}\right)^n$$

(i) Use the Binomial Theorem to find the residue of f at the origin when n is even and when n is odd.

Some relevant facts regarding the binomial theorem (p. 50):

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}\, b^r, \quad \text{where } \binom{n}{r} = \frac{n!}{(n-r)!\,(r!)}$$

Put this into a computer program:

```
Clear["Global`*"]

btf[n_] :=
  (1 / z) Sum[(n! / ((n - r)! r!)) ((z^(n - r)) (1 / z)^r), {r, 0, n}];

Table[{"n=" <> ToString[n], btf[n]} // Expand, {n, 0, 10}]
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Table 1. Binomial theorem applied to f(z) = $\frac{1}{z}\left(z + \frac{1}{z}\right)^n$.

$$\begin{pmatrix}
\text{n=0} & \frac{1}{z} \\
\text{n=1} & 1 + \frac{1}{z^2} \\
\text{n=2} & z + \frac{2}{z} + \frac{1}{z^3} \\
\text{n=3} & z^2 + 3 + \frac{3}{z^2} + \frac{1}{z^4} \\
\text{n=4} & z^3 + 4z + \frac{6}{z} + \frac{4}{z^3} + \frac{1}{z^5} \\
\text{n=5} & z^4 + 5z^2 + 10 + \frac{10}{z^2} + \frac{5}{z^4} + \frac{1}{z^6} \\
\text{n=6} & z^5 + 6z^3 + 15z + \frac{20}{z} + \frac{15}{z^3} + \frac{6}{z^5} + \frac{1}{z^7} \\
\text{n=7} & z^6 + 7z^4 + 21z^2 + 35 + \frac{35}{z^2} + \frac{21}{z^4} + \frac{7}{z^6} + \frac{1}{z^8} \\
\text{n=8} & z^7 + 8z^5 + 28z^3 + 56z + \frac{70}{z} + \frac{56}{z^3} + \frac{28}{z^5} + \frac{8}{z^7} + \frac{1}{z^9} \\
\text{n=9} & z^8 + 9z^6 + 36z^4 + 84z^2 + 126 + \frac{126}{z^2} + \frac{84}{z^4} + \frac{36}{z^6} + \frac{9}{z^8} + \frac{1}{z^{10}} \\
\text{n=10} & z^9 + 10z^7 + 45z^5 + 120z^3 + 210z + \frac{252}{z} + \frac{210}{z^3} + \frac{120}{z^5} + \frac{45}{z^7} + \frac{10}{z^9} + \frac{1}{z^{11}}
\end{pmatrix}$$

We see empirically that the residues for even n are the binomial coefficients. The residues for odd n are 0.

n	Residue	Pos	r	binomial coef
0	1	1	0	0!/((0-0)!0!)
1	0			
2	2	2	1	2!/((2-1)!1!)
3	0			
4	6	3	2	4!/((4-2)!2!)
5	0			
6	20	4	3	6!/((6-3)!3!)
7	0			
8	70	5	4	8!/((8-4)!4!)
9	0			
10	252	6	5	10!/((10-5)!5!)

To explain this, analyze f(z) written as (1/z) times its binomial expression.

$$f(z) = \frac{1}{z}\left(z + \frac{1}{z}\right)^n = \frac{1}{z}\sum_{r=0}^{n}\binom{n}{r} z^{n-r}\left(\frac{1}{z}\right)^r$$

Putting aside the binomial coefficient $\binom{n}{r}$, the core of this for the purpose of determining the power of a summand for any n is

$$\frac{1}{z}\; z^{n-r}\left(\frac{1}{z}\right)^r$$

If this is to equal 1/z, then $z^{n-r}\left(\frac{1}{z}\right)^r$ must equal 1.

$$\frac{z^{n-r}}{z^r} = 1 \implies \text{n-2r = 0, so r = n/2}$$

Since r can only be an integer, n cannot be odd for $\binom{n}{r}$(1/z), so its residue for odd n is 0. We can see from the table that r does indeed equal n/2 for even n. The residue $\binom{n}{r}$ can be rewritten

$$\binom{n}{n/2} = \frac{n!}{(n-n/2)!\,(n/2)!} = \frac{n!}{(n/2)!^2}$$

(ii) If n is odd, what is the value of the integral of f round any loop.

$$\oint_L f(z)dz = 0\; 2\pi\, \text{i}\; v(K, 0) = 0$$

(iii) If n = 2m is even and C is a simple loop winding once round the origin, deduce from part (i) that

$$\oint_C f(z)dz = 2\pi i \frac{(2m)!}{(m!)^2}$$

From (i), the residue is $\frac{n!}{(n/2)!^2}$. Substitute n = 2m:

$$\frac{(2m)!}{(2m/2)!^2} = \frac{(2m)!}{(m)!^2}$$

If C is a simple loop winding once around the origin, the $v(C, 0) = 1$. Then

$$\oint_C f(z)dz = \text{Res } 2\pi \text{ i } v(C, 0) = \frac{(2m)!}{(m!)^2} 2\pi \text{ i}$$

This is what we wanted.

(iv) By taking C to be the unit circle, deduce the following result due to Wallis:

$$\int_0^{2\pi} \cos^{2m}\theta \, d\theta = \frac{(2m)!}{2^{2m-1}(m!)^2}\pi$$

From (i)

$$\oint_C f(z)dz = \text{Res } 2\pi \text{ i } v(C, 0) = \frac{(2m)!}{(m!)^2} 2\pi \text{ i}$$

The equation $f(z) = \frac{1}{z}\left(z + \frac{1}{z}\right)^n$ is carried through from (i), but the integral $\int_0^{2\pi} \cos^{2m}\theta \, d\theta$ is a real integral on the unit circle, so we can not directly apply the principles of complex integration on a loop. Also, the function $\cos^{2m}\theta$ in the problem integral is not equivalent to f(z) in (i). We must first parameterize our complex equation for $\oint_C f(z)dz$ in a way that contains the integral $\int_0^{2\pi} \cos^{2m}\theta \, d\theta$, which we can then isolate and evaluate. Let $z = e^{i\theta}$ (see IX Parametric Evaluation, page 409-410). We seek an integral in the format $\int_a^b f[z(t)]\, v[z(t)]\, dt$, or in this case $\int_0^{2\pi} f[z(\theta)]\, v[z(\theta]\, d\theta$. (We have written v more explicitly than Needham did on p. 409.)

$$v(\theta) = \frac{dz(\theta)}{d\theta} = ie^{i\theta}$$

$$f(z) = \frac{1}{z}\left(z + \frac{1}{z}\right)^n = \frac{1}{e^{i\theta}}\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right)^n$$

$$= e^{-i\theta}\left(e^{i\theta} + e^{-i\theta}\right)^n = 2^n e^{-i\theta} \cos^n\theta$$

From (i) and (ii), n must be even. The problem statement suggests a rewrite in terms of n = 2m.

$$f(z) = 2^{2m} e^{-i\theta} \cos^{2m}\theta$$

Write the parameterized contour integral

$$\oint_C f(z)dz = \int_0^{2\pi} f[z(\theta)]\, v[z(\theta)]\, d\theta$$

$$= \int_0^{2\pi} \left(2^{2m} e^{-i\theta} \cos^{2m}\theta\right) ie^{i\theta}\, d\theta$$

$$= 2^{2m} i \int_0^{2\pi} \cos^{2m}\theta\, d\theta$$

This is a very convenient result. We can write (where $f(z) = \frac{1}{z}\left(z + \frac{1}{z}\right)^{2m}$)

$$2^{2m} i \int_0^{2\pi} \cos^{2m}\theta\, d\theta = \int_0^{2\pi} f[z(\theta)]\, v[z(\theta)]\, d\theta = \oint_C f(z)dz = \frac{(2m)!}{(m!)^2} 2\pi i \qquad \text{[from (iii)]}$$

Then

$$\int_0^{2\pi} \cos^{2m}\theta\, d\theta = \frac{1}{2^{2m} i} \frac{(2m)!}{(m!)^2} 2\pi i = \frac{(2m)!}{2^{2m-1}(m!)^2}\pi$$

This confirms the result of Wallis.

(v) Similarly, by considering functions of the form z^k f(z) where k is an integer, evaluate

$$\int_0^{2\pi} \cos^n\theta \cos k\theta\, d\theta \quad \text{and} \quad \int_0^{2\pi} \cos^n\theta \cdot \sin k\theta\, d\theta$$

The procedure is to first evaluate the contour integral using the binomial expansion to find the residue which is multiplied by the RHS of (6) from p. 392. Then the form z^k *f(z)* is parameterized and the integral is evaluated. This result is set equal to the result of the contour integral. Then equate real and imaginary parts and solve for the desired integrals.

Evaluate the contour integral:

$$z^k f(z) = z^k \left(\frac{1}{z}\right)\left(z + \frac{1}{z}\right)^{2m} = z^{k-1} \binom{n}{r} z^{n-r} \left(\frac{1}{z}\right)^r$$

In order for there to be a non-zero integral

$$z^{k-1+n-r}\left(\frac{1}{z}\right)^r = \frac{1}{z}$$

Write the equation of powers and solve for r.

$$k-1+n-2r = -1$$

$r = (n+k)/2$

We know that n must be even and we can see from the factorials of $\frac{(n+k)}{2}$, that (n+k) must also be even, so k must be even. Only functions with this combination have a 1/z summand in the binomial expansion.

Write the equation for the residue

$$\text{Res} = \binom{n}{\frac{(n+k)}{2}} = \frac{n!}{\left(n-\frac{(n+k)}{2}\right)!\left(\frac{(n+k)}{2}\right)!} = \frac{n!}{\left(\frac{(n-k)}{2}\right)!\left(\frac{(n+k)}{2}\right)!}$$

Then the contour integral is

$$\oint_C z^k f(z)\, dz = \text{Res} \cdot 2\pi i\, v(C, 0) = \frac{n!}{\left(\frac{(n-k)}{2}\right)!\left(\frac{(n+k)}{2}\right)!} 2\pi\, i$$

Parameterize the contour integral:

$$z = e^{i\theta},\ v = ie^{i\theta}$$

$$z^k\left(\frac{1}{z}\right)\left(z + \frac{1}{z}\right)^n = e^{ik\theta}\left(e^{i\theta} + e^{-i\theta}\right)^n = 2^n\, e^{ik\theta}\, e^{-i\theta} \cos^n\theta$$

Then

$$\int_0^{2\pi}\left(2^n\, e^{ik\theta}\, e^{-i\theta} \cos^n\theta\right) ie^{i\theta}\, d\theta = 2^n i \int_0^{2\pi} e^{ik\theta} \cos^n\theta\, d\theta$$

$$= 2^n i \int_0^{2\pi} (\cos k\theta + i\sin k\theta) \cos^n\theta\, d\theta$$

$$= 2^n i \int_0^{2\pi} \cos^n\theta \cos k\theta\, d\theta - 2^n \int_0^{2\pi} \cos^n\theta \sin k\theta\, d\theta = \frac{n!}{\left(\frac{(n-k)}{2}\right)!\left(\frac{(n+k)}{2}\right)!} 2\pi i$$

because the parameterized integral must equal the contour integral.

Equate real and imaginary parts. From the RHS, we deduce that real part is 0.

$$2^n i \int_0^{2\pi} \cos^n\theta \cos k\theta\, d\theta = \frac{n!}{\left(\frac{(n-k)}{2}\right)!\left(\frac{(n+k)}{2}\right)!} 2\pi i$$

The result is

$$\int_0^{2\pi} \cos^n\theta \cos k\theta\, d\theta = \frac{n!}{2^{n-1}\left(\frac{(n-k)}{2}\right)!\left(\frac{(n+k)}{2}\right)!} \pi$$

for n and k even. If n or k odd, the integral is 0.

$$\int_0^{2\pi} \cos^n\theta \sin k\theta\, d\theta = 0$$

for all n and k.