Needham, Chapter 8, Exercise 16, p. 423.
G. Palmer, July 23, 2017, 6:30 am PST

*16 (i) Show that when integrating a product of analytic functions, we may use the ordinary method of "integration by parts."*

Let f(z) and g(z) be complex analytic functions. Then we want to show that integration by parts can be used to integrate fg.

Use $\int_a^b u\,dv$ = $[uv]_a^b$ - $\int_a^b v\,du$.

Since f and g are both analytic, the composition f(z) g(z) is also analytic.

Let u = f(z), and let v = $\int$ g(z) dz = $\frac{g^2(z)}{2}$. Then du = f'(z) dz and dv = g(z) dz.

$$\int f(z)g(z)dz = [f(z)\ \tfrac{g^2(z)}{2}]_a^b - \int \tfrac{g^2(z)}{2}\ f'(z)\ dz \qquad (1)\ \int udv = uv - \int vdu$$

Now expand the second integral on the RHS to show that the result is $\int$ f(z) g(z) dz.

Now let u = $\frac{g^2(z)}{2}$ and let v = f(z). Then du = g(z) dz

$$\int \tfrac{g^2(z)}{2}\ f'(z)\ dz = \left[\tfrac{g^2(z)}{2}\ f(z)\right]_a^b - \int f(z)\ g(z)\ dz \qquad (2)\ \int udv = uv - \int vdu$$

Substitute (2) into (1)

$$\int f(z)g(z)dz = [\ \tfrac{g^2(z)}{2}\ f(z)]_a^b - [[\ \tfrac{g^2(z)}{2}\ f(z)]_a^b - \int f(z)\ g(z)\ dz]$$

$$\int f(z)g(z)dz = \int f(z)\ g(z)\ dz \qquad \text{This is what we wanted.}$$

We try this another way:

Let u = f(z), v = $\int$ g(z) dz. Then du = f'(z) dz, dv = g(z).

$$\int f(z)\ g(z)\ dz = f(z) \int g(z)\ dz - \int(\int g(z)\ dz)\ f'(z)\ dz \qquad (1)$$

Now expand the second integral on the RHS to show that the result is $\int$ f(z) g(z) dz.

Let u = $\int$ g(z) dz, v = f(z). Then du = g(z) dz.

$$\int(\int g(z)\ dz)\ f'(z)\ dz = f(z) \int g(z)\ dz - \int f(z)\ g(z)\ dz \qquad (2)$$

$\int f(z)\, g(z)\, dz = f(z) \int g(z)\, dz - ( f(z) \int g(z)\, dz - \int f(z)\, g(z)\, dz)$ by substitution of RHS of (2) into (1).

$\int f(z)\, g(z)\, dz = \int f(z)\, g(z)\, dz$ This is what we wanted.

*(ii) Let L be a contour from the real number* $-\theta$ *to* $+\theta$*. Show that*

$$\int_L z e^{iz}\, dz = 2i\,(\sin\theta - \theta\cos\theta)$$

Answer. Note that we find the integral in terms of z. The contour begins and ends at the real numbers $-\theta$ to $\theta$, but the real line is not necessarily traversed by z. The contour might lie off the real line, or it might cross and recross the real line. The integral is path independent. There is no need to write $z = re^{i\theta}$ or to find a function $z(\theta)$. One simply does the integration by parts and substitutes $-\theta$ or $\theta$ for z.

Rewrite $e^{iz}$ as (cos z + i sin z).

$$\int_L z(\cos z + i\sin z)\, dz = \int_L z\cos z\, dz + i\int_L z\sin z\, dz$$

Find both the integrals on the RHS using integration by parts.

Let u = z and v = sin z. Then du = 1 and dv = cos z.

$$\int_L z\cos z\, dz = z\sin z - \int_L \sin z\, dz \qquad [\textstyle\int udv = uv - \int vdu]$$

$$= [z\sin z - (-\cos z)]_L = [z\sin z + \cos z]_L, \text{ noting that } L = [-\theta..\theta]$$

Let u = z and v = -cos z. Then du = 1 and dv = sin z

$$\int_L z\sin z\, dz = [z(-\cos z)]_L - \int_L (-\cos z)\, dz] = [-z\cos z]_L + \int_L \cos z\, dz]$$

$$= [-z\cos z + \sin z]_L$$

$$\int_L z e^{iz}\, dz = [z\sin z + \cos z]_L + i\,[-z\cos z + \sin z]_L$$

$$= [z\sin z + \cos z - i\,z\cos z + i\sin z)]_{-\theta}^{\theta}$$

$$= \theta\sin\theta + \cos\theta - i\,\theta\cos\theta + i\sin\theta - (-\theta\sin(-\theta) + \cos(-\theta) - i\,(-\theta)\cos(-\theta) + i\sin(-\theta))$$

$$= \theta\sin\theta + \cos\theta - i\,\theta\cos\theta + i\sin\theta - (\theta\sin\theta + \cos\theta + i\,\theta\cos\theta - i\sin\theta)$$

$$= \theta\sin\theta + \cos\theta - i\,\theta\cos\theta + i\sin\theta - \theta\sin\theta - \cos\theta - i\,\theta\cos\theta + i\sin\theta$$

$= (\theta \sin\theta - \theta \sin\theta) + (\cos\theta - \cos\theta) - i\,\theta\cos\theta - i\,\theta\cos\theta + (i\sin\theta + i\sin\theta))$

$= 2i(\sin\theta - \theta\cos\theta)$

Note: To avoid all the bookkeeping of signs, it is better to integrate $\int_L z e^{iz}\, dz$ without rewriting $z e^{iz}$.

$\int_L z e^{iz}\, dz = \int_{-\theta}^{\theta} z e^{iz}\, dz$

Let u = z, v = $\frac{e^{iz}}{i}$. Then du = dz, dv = $e^{iz}$dz.

$\int_{-\theta}^{\theta} z e^{iz}\, dz = [z\,\frac{e^{iz}}{i}]_{-\theta}^{\theta} - \int_{-\theta}^{\theta} \frac{e^{iz}}{i} dz$

$= [z\,\frac{e^{iz}}{i} - \frac{1}{i}\,\frac{e^{iz}}{i}]_{-\theta}^{\theta} = [-z\,e^{iz} + e^{iz}]_{-\theta}^{\theta}$

$= -i\,\theta\, e^{i\theta} + e^{i\theta} - (-i\,(-\theta)\, e^{-i\theta} + e^{-i\theta}) = -i\,\theta\, e^{i\theta} - i\theta\, e^{-i\theta} + e^{i\theta} + e^{-i\theta})$

$= -i\,\theta\left(e^{i\theta} + e^{-i\theta}\right) + e^{i\theta} - e^{-i\theta} = -2\,i\,\theta\cos\theta + 2i\sin\theta$

$= 2i(\sin\theta - \theta\cos\theta)$