Needham, Chapter 8, Exercise 15, p. 422-423.
G. Palmer, July 20, 2017, 6 pm PST

15 (i) Use the fundamental Theorem to write down the value of

$$\int_0^{a+ib} e^z \, dz$$

$$\int_0^{a+ib} e^z \, dz = e^z \,\Big|_0^{a+ib} = e^{a+ib} - 1$$

$$= e^a e^{ib} - 1 = e^a(\cos b + i \sin b) - 1$$

$$= (e^a \cos b - 1) + e^a i \sin b \qquad \text{(1) dividing into real and imaginary terms}$$

(ii) Equate the answer with the one obtained by parametric evaluation along the straight contour from 0 to (a+ib), and deduce that

$$\int_0^1 e^{ax} \cos bx \, dx = (a(e^a \cos b - 1) + b e^a \sin b) / (a^2 + b^2)$$

and

$$\int_0^1 e^{ax} \sin bx \, dx = (b(1 - e^a \cos b) + a e^a \sin b) / (a^2 + b^2)$$

[*Mathematica*'s editor seems incapable of typesetting this properly.]

Parametric evaluation of $\int_0^{a+ib} e^z \, dz$:

$$z(x) = (a+ib)\, x, \qquad x \text{ in } [0..1]$$

$$v(x) = a + ib$$

$$\int_0^1 z(x)\, v(x)\, dx = \int_0^1 e^{(a+ib)\,x} (a + ib)\, dx$$

$$= (a + ib) \int_0^1 e^{ax} (\cos bx + i \sin bx) dx$$

$$= (a + ib) \int_0^1 e^{ax} \cos bx \, dx + (a + ib) \int_0^1 e^{ax}\, i \sin bx \, dx$$

Equate answer by Fundamental Theorem with answer by parametric evaluation

$$(e^a \cos b - 1) + e^a i \sin b = (a + ib) \int_0^1 e^{ax} \cos bx \, dx + (a + ib) \int_0^1 e^{ax}\, i \sin bx \, dx$$

Multiply through by (a-ib):

$$(a-ib)((e^a \cos b - 1) + e^a\, i \sin b) = \left(a^2+b^2\right)\int_0^1 e^{ax}\cos bx\, dx + \left(a^2+b^2\right)\int_0^1 e^{ax}\, i \sin bx\, dx$$

$$(a\,(e^a \cos b - 1) + b\, e^a \sin b) - i\,(b\,(e^a \cos b - 1) - a\, e^a \sin b) =$$
$$\left(a^2+b^2\right)\int_0^1 e^{ax}\cos bx\, dx + \left(a^2+b^2\right)\int_0^1 e^{ax}\, i \sin bx\, dx \qquad (2)$$

$$\left(a^2+b^2\right)\int_0^1 e^{ax}\cos bx\, dx = (a\,(e^a \cos b - 1) + b\, e^a \sin b)$$ [taking real parts of both sides of (2)]

$$\int_0^1 e^{ax}\cos bx\, dx = \frac{(a\,(e^a \cos b-1)+b\, e^a \sin b)}{a^2+b^2}$$ This is what we wanted for the first equation.

$$-\ (b\,(e^a \cos b - 1) - a\, e^a \sin b) = \left(a^2+b^2\right)\int_0^1 e^{ax} \sin bx\, dx$$ [taking imaginary parts of both sides of (2)]

$$\int_0^1 e^{ax} \sin bx\, dx = \frac{b\,(1-e^a \cos b)+a\, e^a \sin b}{a^2+b^2}$$ This is what we wanted for the second equation.

(iii) Prove the results in (ii) by ordinary methods.

Find $\int e^{ax}\cos bx\, dx$ and $\int e^{ax}\, i \sin bx\, dx$ substituting $e^{(a+bi)\,x}$ and integrating. Assume that the constant of integration is 0. We use this method in preference to integration by parts.

$$\int e^{ax}\cos bx\, dx + i\int e^{ax}\sin bx\, dx = \int e^{ax}(\cos bx + i \sin bx)\, dx = \int e^{(a+bi)\,x} dx$$

$$= \frac{e^{(a+bi)\,x}}{a+bi} = \frac{a-bi}{a-bi}\,\frac{e^{(a+bi)\,x}}{a+bi} = \frac{(a-bi)\, e^{(a+bi)\,x}}{a^2+b^2} = \frac{e^{ax}(a-bi)\,(\cos bx + i \sin bx)}{a^2+b^2}$$

$$= \frac{e^{ax}(a-bi)\,(\cos bx + i \sin bx)}{a^2+b^2} = \frac{e^{ax}[(a \cos bx + a\, i \sin bx) + (-bi \cos bx + b \sin bx)]}{a^2+b^2}$$

Group real and imaginary parts of numerator.

$$= \frac{e^{ax}(a \cos bx + b \sin bx) + (a\, i \sin bx - bi \cos bx\,)}{a^2+b^2}$$

$$= \frac{e^{ax}(a \cos bx + b \sin bx\,)}{a^2+b^2} + i\, \frac{e^{ax}(\, a \sin bx - b \cos bx\,)}{a^2+b^2}$$

Equate real and imaginary parts

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx\,)}{a^2+b^2}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}$$

Evaluate the results on 0 to 1.

$$\int_0^1 e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2} \Big|_0^1 = \frac{e^a(a \cos b + b \sin b)}{a^2+b^2} - \frac{e^0(a \cos 0 + b \sin 0)}{a^2+b^2}$$

$$= \frac{e^a(a \cos b + b \sin b) - a}{a^2+b^2} = \frac{e^a\, a(\cos b - 1) + b\, e^a \sin b}{a^2+b^2}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} \Big|_0^1 = \frac{e^a(a \sin b - b \cos b)}{a^2+b^2} - \frac{e^0(a \sin 0 - b \cos 0)}{a^2+b^2}$$

$$= \frac{e^a(a \sin b - b \cos b) + b}{a^2+b^2} = \frac{e^a(a \sin b - b \cos b) + b}{a^2+b^2} = \frac{b(1 - e^a \cos b) + a\, e^a \sin b}{a^2+b^2}$$

Vasco wrote “Your method uses complex analysis and seems to me to be another way of writing the method of part (ii)”, so I will try using only integration by parts:

Deduce that

$$\int_0^1 e^{ax} \cos bx \, dx = (a(e^a \cos b - 1) + b e^a \sin b) \big/ (a^2 + b^2)$$

Let $u = \cos bx$ and $v = \frac{e^{ax}}{a}$. Then du = $b \sin x$ and dv = e^{ax}.

$$\int_0^1 e^{ax} \cos bx \, dx = \int_0^1 \cos bx \, e^{ax} \, dx \qquad [\textstyle\int udv]$$

$$= \left[\cos bx \, \frac{e^{ax}}{a}\right]_0^1 + \frac{b}{a}\int_0^1 e^{ax} \sin bx \, dx \qquad (1)\ [uv] - \textstyle\int vdu$$

Integrate $\int_0^1 e^{ax}$ *sin bx dx: Let u = sin bx and v =* $\frac{e^{ax}}{a}$*. Then du = b cos bx and dv =* e^{ax}.

$$\int_0^1 e^{ax} \sin bx \, dx = \int_0^1 \sin bx \, e^{ax} \, dx \qquad [\textstyle\int udv]$$

$$= \left[\sin bx \, \frac{e^{ax}}{a}\right]_0^1 - \frac{b}{a}\int_0^1 e^{ax} \cos bx \, dx \qquad (2)\ [uv] - \textstyle\int vdu$$

$$\frac{b}{a}\int_0^1 e^{ax} \cos bx \, dx = \left[\sin bx \, \frac{e^{ax}}{a}\right]_0^1 - \int_0^1 e^{ax} \sin bx \, dx \qquad \text{(by rearranging)}$$

$$\int_0^1 e^{ax} \cos bx \, dx = \frac{a}{b}\left(\left[\sin bx \, \frac{e^{ax}}{a}\right]_0^1 - \int_0^1 e^{ax} \sin bx \, dx\right) \qquad \text{(3) (to be used in 2nd part)}$$

$$\int_0^1 e^{ax} \cos bx \, dx = \left[\cos bx \, \frac{e^{ax}}{a}\right]_0^1 + \frac{b}{a}\left(\left[\sin bx \, \frac{e^{ax}}{a}\right]_0^1 - \frac{b}{a}\int_0^1 e^{ax} \cos bx \, dx\right) \quad \text{[by substit. (2) into (1).}$$

$$\int_0^1 e^{ax}\cos bx\, dx + \frac{b^2}{a^2}\int_0^1 e^{ax}\ \cos bx\, dx\ =\ \left[\cos bx\ \frac{e^{ax}}{a}\right]_0^1 + \frac{b}{a}\left[\ \sin bx\ \frac{e^{ax}}{a}\right]_0^1$$

$$\int_0^1 e^{ax}\cos bx\, dx = \frac{1}{\left(1+\frac{b^2}{a^2}\right)}\ \left(\ \left[\cos bx\ \frac{e^{ax}}{a}\right]_0^1 + \frac{b}{a}\left[\ \sin bx\ \frac{e^{ax}}{a}\right]_0^1\right)$$

$$\int_0^1 e^{ax}\cos bx\, dx = \frac{1}{\left(1+\frac{b^2}{a^2}\right)}\ \left(\cos b\left(\ \frac{e^a}{a}\right) - \ \frac{1}{a}\right) + \frac{b}{a}\ \sin b\ \left(\frac{e^a}{a}\right))$$

$$= \frac{a^2}{a^2\left(1+\frac{b^2}{a^2}\right)}\ (\ (\cos b\left(\ \frac{e^a}{a}\right) - \ \frac{1}{a}) + \frac{b}{a}\ \sin b\ \left(\frac{e^a}{a}\right))$$

$$= \frac{a\,(e^a \cos b\ - 1)+b\, e^a \sin b}{(a^2+b^2)}$$ This is what we wanted.

And deduce that

$$\int_0^1 e^{ax}\sin bx\, dx = (b(1 - e^a \cos b)\ +\ a\, e^a \sin b)\big/\left(a^2 + b^2\right)$$

Then from substitution of the previous result into (3) :

$$\frac{ae^a \cos b\ - 1)+b\, e^a \sin b}{(a^2+b^2)}\ =\ \frac{a}{b}(\ \left[\sin bx\ \frac{e^{ax}}{a}\right]_0^1 - \int_0^1 e^{ax}\ \ \sin bx\, dx)$$

$$\frac{a}{b}\int_0^1 e^{ax}\ \ \sin bx\, dx) = \frac{a}{b}(\ \left[\sin bx\ \frac{e^{ax}}{a}\right]_0^1 - \frac{a\,(e^a \cos b\ - 1)+b\, e^a \sin b}{(a^2+b^2)}$$

$$\int_0^1 e^{ax}\ \ \sin bx\, dx = \frac{b}{a}\left(\frac{a}{b}\sin b\ \frac{e^a}{a} - \frac{a\,(e^a \cos b\ - 1)+b\, e^a \sin b}{(a^2+b^2)}\right)$$

$$= \frac{(a^2+b^2)}{(a^2+b^2)}\frac{e^a}{a}\sin b\ \ - \frac{b}{a}\frac{a\,(e^a \cos b\ - 1)+b\, e^a \sin b}{(a^2+b^2)}$$

$$= \frac{(a^2+b^2)\, e^a \sin b - b(a\,(e^a \cos b\ - 1) - b^2\, e^a \sin b}{(a^2+b^2)\, a}$$

$$= \frac{a^2\, e^a \sin b - b(a\,(1 - ae^a \cos b)}{(a^2+b^2)\, a}$$ add like terms and rearrange signs

$$= \frac{b(1 - ae^a \cos b) + a^2\, e^a \sin b}{(a^2+b^2)}$$ cancel a in numerator and denominator