Needham, Chapter 8, Exercise 14, p. 422.
G. Palmer, July 16, 2017, 1 pm PST

14 (i) The integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)}$$

is easily found by ordinary means, but evaluate it instead by the method of the previous exercise.

Evaluate by fundamental theorem on trig equivalent:

$$\int_{-\infty}^{\infty} \mathrm{dx}/\left(x^2+1\right) = \arctan(\mathrm{x})\,|_{-\infty}^{\infty} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$$

Evaluate by decomposition and (7):

$$1/\left(x^2+1\right) = \frac{1}{(x-i)\,(x+i)} = \frac{A}{(x-i)} + \frac{B}{(x+i)}$$

1 = A(x+i) + B(x-i)

$\Longrightarrow$ B = -A and 1 = Ai -Bi

1 = 2Ai

$A = \frac{1}{2i}$ $B = -\frac{1}{2i}$

$$\int_{L+J} 1/\left(x^2+1\right) = \frac{1}{2i}\int_{L+J}\left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]\mathrm{dx}$$

There are two poles, i and -i. Making R > 1, we can write

$$\frac{1}{2i}\int_{L+J}\left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]\mathrm{dx} = 2\pi\mathrm{i}\frac{1}{2i}[\nu(\mathrm{L+J,i}) - \nu(\mathrm{L+J,-i})]$$

$$= \pi\,[1 + 0] = \pi$$

We use inequality (5) to show that the integral round J dies away to zero as R goes to infinity:

$$\frac{\operatorname{len} J}{M} \sim \pi\mathrm{R}/\left|\, 1 - R^2 \,\right|$$

$$\lim_{R\to\infty} \left|\int_J \mathrm{dz}/\left(z^2+1\right)\right| \le \lim_{R\to\infty} \pi\mathrm{R}/\left|\,1-R^2\,\right| = \lim_{R\to\infty} \pi/R = 0$$

Deduce the value of $\int_{-\infty}^{\infty} \mathrm{dx}/\left(x^2+1\right)$

$$\int_{-\infty}^{\infty} \mathrm{dx}/\left(x^2+1\right) = \int_{L+J} \mathrm{dx}/\left(x^2+1\right) - \int_J \mathrm{dx}/\left(x^2+1\right) = \pi - 0 = \pi$$

The integral obtained by decomposition equals that obtained by the fundamental theorem and trig substitution.

(ii) Likewise, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^2}$$

by ordinary means and then by contour integration. [Hint: The quickest way to find the partial fraction decomposition for this function is to square the decomposition of $1/\left(z^2 + 1\right)$.]

By ordinary differentiation we get

$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{1}{2}\left(x/\left(1+x^2\right) + \arctan(x)\right)_{-\infty}^{\infty} \quad (\textit{Mathematica})$$

$$= \frac{1}{2}\left(0 + \frac{\pi}{2} - 0 - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}$$

By decomposition of $1/\left(z^2 + 1\right)$:

$$f(z) = 1/\left(z^2+1\right) = \frac{1}{2i}\left(\frac{1}{(z-i)} - \frac{1}{(z+i)}\right)$$

Square the decomposition:

$$\left(\frac{1}{2i}\left(\frac{1}{(z-i)} - \frac{1}{(z+i)}\right)\right)^2 = -\left(\frac{1}{4}\right)\left[1/(z-i)^2 - 2\frac{1}{(z-i)}\frac{1}{(z+i)} + 1/(z+i)^2\right]$$

In § 3, subsection 2 Complex Inversion as a Limiting Case, p. 397, Needham stated *for a complex circular loop, there is a striking and fundamental difference between complex inversion and all other powers: if m ≠ -1 then the integral vanishes*. Only the decomposition terms of $2\frac{1}{(z-i)}\frac{1}{(z+i)}$, have power -1. The sqared terms inside the square brackets have powers m = -2. From 13(i), we can deduce that

$$2\frac{1}{(z-i)}\frac{1}{(z+i)} = \frac{2}{2i}\left(\frac{1}{(z-i)} - \frac{1}{(z+i)}\right) = -i\left(\frac{1}{(z-i)} - \frac{1}{(z+i)}\right)$$

Then

$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^2} = \frac{1}{4}(-i)\int_{L+J}\left(\frac{1}{(z-i)} - \frac{1}{(z+i)}\right) dz$$

$$= 2\pi i \left(\frac{-i}{4}\right)\left[v(L+J,i) - v(L+J,-i)\right] = \frac{\pi}{2}(1-0) = \frac{\pi}{2}$$

We use inequality (5) to show that the integral round J dies away to zero as R goes to infinity:

$$\frac{\text{len}\,J}{M} \sim \pi\text{R} / \mid 1 - R^2 \mid\ ^2$$

$$\lim_{R\to\infty} \left| \int_J \text{dz} \big/ \left(z^2 + 1\right)^2 \right| \leq \lim_{R\to\infty} \pi\text{R} / \mid 1 - R^2 \mid\ ^2 = \lim_{R\to\infty} \pi \big/ R^3 = 0$$

Deduce the value of $\int_{-\infty}^{\infty} \text{dx} \big/ \left(x^2 + 1\right)^2$

$$\int_{-\infty}^{\infty} \text{dx} \big/ \left(x^2 + 1\right)^2 = \int_{L+J} \text{dx} \big/ \left(x^2 + 1\right)^2 - \int_J \text{dx} \big/ \left(x^2 + 1\right)^2 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

This equals the integration by ordinary means.