Needham, Chapter 8, Exercise 13, p. 421-422.
G. Palmer, July 17, 2017, 7 am PST

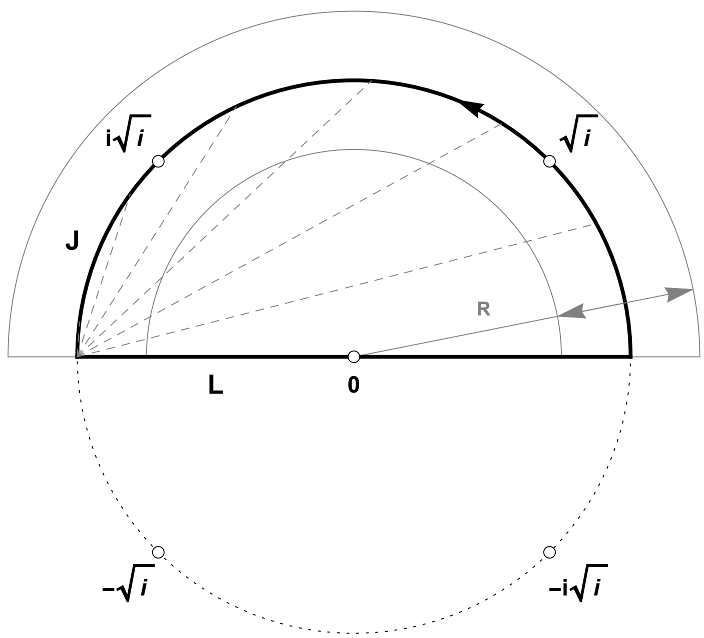

Figure 1. Loop on half disc

13 This exercise illustrates how one type of difficult real integral may be evaluated easily using a complex integral. Let L be the straight contour along the real axis from -R to + R, and let J be the semi-circular contour (in the upper half plane) back from +R to -R. The complete contour L + J is thus a closed loop.

(i) Using the partial fraction idea of the previous exercise, show that the integral

$$\oint_{L+J} \frac{dz}{(z^4+1)}$$

vanishes if R < 1, and find its value if R > 1.

$$\oint_{L+J} dz/(z^4+1) = \oint_{L+J} dz/((z^2-i)(z^2-i))$$

Find partial decomposition:

$$\frac{1}{z^4+1} = \frac{A}{z^2-i} + \frac{B}{z^2+i} \qquad (1)$$

$$1 = A(z^2+i) + B(z^2-i)$$

$$Az^2 + Bz^2 = 0 \implies A = -B \qquad (2)$$

$1 = Ai - Bi$

$= 2Ai$ by substitution from (2)

$A = \frac{1}{2i}$, $B = -\frac{1}{2i}$

$1/(z^4+1) = \frac{1}{2i}\left(1/(z^2-i) - 1/(z^2+i)\right)$ by substitution into (1)

Find partial decomposition of terms:

$$\frac{1}{z^2-i} = \frac{C}{z-\sqrt{i}} + \frac{D}{z+\sqrt{i}}$$

$1 = C(z+\sqrt{i}\,) + D(z-\sqrt{i}\,)$

$C = -D$

$1 = C\sqrt{i} - D\sqrt{i} = 2C\sqrt{i}$

$C = \frac{1}{2\sqrt{i}}$, $D = -\frac{1}{2\sqrt{i}}$

$1/(z^2-i) = \frac{1}{2\sqrt{i}}\left(\frac{1}{z-\sqrt{i}} - \frac{1}{z+\sqrt{i}}\right)$

$$\frac{1}{z^2+i} = \frac{E}{z-i\sqrt{i}} + \frac{F}{z+i\sqrt{i}}$$

$1 = E(z+i\sqrt{i}\,) + F(z-i\sqrt{i}\,)$

$E = -F$

$1 = E\,i\sqrt{i} - F\,i\sqrt{i}$

$= 2E\,i\sqrt{i}$

$E = \frac{1}{2i\sqrt{i}}$, $F = -\frac{1}{2i\sqrt{i}}$

$1/(z^2+i) = \frac{1}{2i\sqrt{i}}\left(\frac{1}{z-i\sqrt{i}} - \frac{1}{z+i\sqrt{i}}\right)$

Sum the integrals for the parts, remembering that the second term of the integral is negative .

$$\oint_{L+J} dz/(z^4+1) = \frac{1}{2i}\frac{1}{2\sqrt{i}}\oint_{L+J}\left(\frac{1}{z-\sqrt{i}} - \frac{1}{z+\sqrt{i}}\right) - \frac{1}{i}\left(\frac{1}{z-i\sqrt{i}} - \frac{1}{z+i\sqrt{i}}\right)dz$$

$$= \frac{1}{4i\sqrt{i}} \oint_{L+J}\left(\frac{1}{z-\sqrt{i}} - \frac{1}{z+\sqrt{i}} + i\frac{1}{z-i\sqrt{i}} - i\,\frac{1}{z+i\sqrt{i}}\right)dz$$

The singularities are $\sqrt{i}$, $-\sqrt{i}$, $i\sqrt{i}$, $-i\sqrt{i}$ (Figure 1). These singularities all fall on the unit circle. If R < 1, the loop L+J contains no singularities of $f(z) = 1/(z^4 + 1)$. Since f is analytic, its integral disappears on the small contour. If R > 1, the singularities $\sqrt{i}$ and $i\sqrt{i}$ fall within the loop L+J, each with v[L+J, p] = 1. Notice that these are the two positive singularities. I at first thought that the windings should be negative because the the windings of analytic functions around poles have minus signs. But the windings that are counted by (7) are the windings of the base loop itself. Hence, the windings of the closed loop (L+J) are positive.

$$\oint_{L+J} dz/(z^4 + 1) = 2\pi\,i\,\left(\frac{1}{4i\sqrt{i}}[v(L+J, \sqrt{i}) - (v(L+J, -\sqrt{i}) + iv(L+J, i\sqrt{i}) - iv(L+J, -i\sqrt{i})]\right. \qquad \text{by (7)}$$

$$= \frac{\pi}{2\sqrt{i}}[1 - 0 + i(1) - 0] = \frac{\pi}{2\sqrt{i}}(1+i)$$

Since $\sqrt{i} = e^{i\pi/4}$ and $(1+i) = \sqrt{2}\, e^{i\pi/4}$

$$\frac{\pi}{2\sqrt{i}}(1\text{-}i) = \frac{\sqrt{2}\,\pi}{2}\,\frac{e^{i\pi/4}}{e^{i\pi/4}} = \frac{\sqrt{2}\,\pi}{2}$$

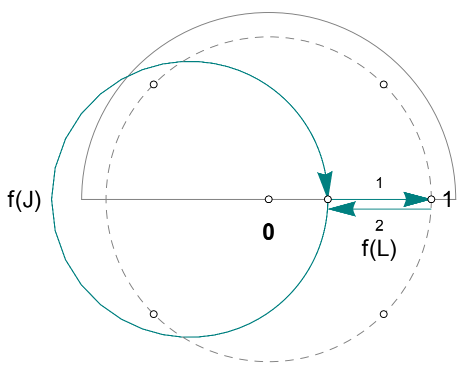

Figure 2. $f=1/(z^4+1)$ on half disc loop
R=1.15

Figure 2 shows the traversal of f on the rim L+J of the half disc. On one counterclockwise traversal of J alone with R > 1, f makes two clockwise winds around the origin, one for each of the poles $\sqrt{i}$ and $i\sqrt{i}$. On the counterclockwise (-R to R) traversal of L, f travels along the real line from f(-R) to 1, and back from 1 to f(R), and f(-R) = f(R). The total winding of f on L around any pole or the origin must be 0.

$$v(f(L+J,0) = v(f(L), 0) + v(f(J), 0) = 0 - 2 = -2$$

These facts are points of interest only, not pursuant to the question.

(ii) Using the fact that $z^4 + 1$ is the complex number from -1 to z^4, write down the minimum value of $|z^4+1|$ as z travels round J. Now think of R as large, and use inequality (5) to show that the integral round J dies away to zero as R grows to infinity.

In Figure 1, the gray dashed lines are lines from -1 to z^4 as z traverses J with R = 1. The plot shows that z^4 is closest to -1 when $z^4 = -R^4$. This is true whether R is larger or smaller than 1. When R = 1, the distance is zero. Then the minimum distance from -1 to z^4 and from the origin to the image curve (Figure 2) equals $|-R^4-(-1)| = |1 - R^4|$. The maximum distance from -1 to z^4 of R^4+1 occurs where $z^4 = R^4$.

Inequality (5): $\left|\int_K f(z)\,\mathrm{dz}\right| \leq$ M · (length of K), where M denotes the maximum distance from the origin to the image curve. M is the reciprocal of the minimum distance.

$$\left|\int_J \left(1/\left(z^4+1\right)\right)\mathrm{dz}\right| \leq 1/\left|1-R^4\right| \cdot \text{(length of J)}$$

The length of J is πR. As R grows to infinity, the length of J also grows to infinity in proportion to R, but M shrinks in proportion to $1/R^4$, and the absolute value of the integral shrinks in proportion to $1/R^3$.

$$\frac{\text{len}\,J}{M} \sim \pi\mathrm{R}/\left|1-R^4\right|$$

$$\lim_{R\to\infty}\left|\int_J \mathrm{dz}/\left(z^4+1\right)\right| \leq \lim_{R\to\infty}\pi\mathrm{R}/\left|1-R^4\right| = \pi/R^3 = 0,$$

which is one would expect by inspecting an approximation of the integral: $\left|\int z^{-4}\,dz\right| = |\frac{1}{3}z^{-3}|$.

(iii) From the previous parts, deduce the value of

$$\int_{-\infty}^{\infty} dx/\left(x^4+1\right).$$

Part (ii) showed that the integral on J alone disappears as R $\longrightarrow \infty$. We also know that the integral on the whole loop L+J is $\frac{\sqrt{2}\,\pi}{2}$. Then we need only the integral on L, which must be the difference between the integral on L+J and the integral on J. Since the integral on -∞ to ∞ must be real, we take only the real part.

$$\int_{-\infty}^{\infty} \mathrm{dx}/\left(x^4+1\right) = -\frac{\sqrt{2}\,\pi}{2}$$