Needham, Chapter 8, Exercise 12, p. 421.
G. Palmer, July 11, 2017, 5:10 pm PST

12 Let K be the contour in [21].

(i) Evaluate the following integral by factoring the denominator and putting the integrand into partial fractions.

$$\oint_K \left(\frac{z}{z^2 - iz - 1 - i}\right) dz$$

$$\frac{z}{z^2 - iz - 1 - i} = \frac{1}{5}\left[\frac{3+i}{(z-1-i)} + \frac{2-i}{z+1}\right]$$

$$\oint_K \left(\frac{z}{z^2 - iz - 1 - i}\right) dz = \frac{1}{5}\oint_K \left(\frac{3+i}{(z-1-i)} + \frac{2-i}{z+1}\right) dz$$

The partial functions have singularities at (1+i) and -1. Then by (7),

$$\oint_K \left(\frac{z}{z^2 - iz - 1 - i}\right) dz = \frac{2\pi i}{5}\ [\ (3+i)\ \nu[K,\ 1+i] + (2 - i)\ \nu[K,\ -1]]$$

From [21], it appears that $\nu[K, 1+i] = 0$, and $\nu[K, -1] = -2$. Then

$$\oint_K \left(\frac{z}{z^2 - iz - 1 - i}\right) dz = \frac{2\pi i}{5}\ [\ (3+i)\ 0 + (2 - i)\ (-2)]$$

$$= \frac{2\pi i}{5}\ (-4 + 2i) = \frac{4\pi i}{5}(-2+i)$$

Multiplying through by $\frac{2+i}{2+i}$, we get $\frac{-4\pi i}{2+i}$, which is what Vasco has.

(ii) Write down the Laurent series (centred at the origin) for ($\cos z/z^{11}$). Hence find

$$\oint \frac{\cos z}{z^{11}} dz.$$

cos z can be rewritten as the series

$$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

($\cos z/z^{11}$) has the Laurent series

$$\frac{1}{z^{11}} - \frac{1}{2!\,z^9} + \frac{1}{4!\,z^7} - \frac{1}{6!\,z^5} + \frac{1}{8!\,z^3} - \frac{1}{10!\,z} + \ldots$$

so $\frac{\cos z}{z^{11}}$ has a complex inversion term $-\frac{1}{10!\,z}$. All the other terms have zero integrals. The residue is $(-\frac{1}{10!})$.
Then,

$$\oint \frac{\cos z}{z^{11}} \mathrm{d}z = -\frac{1}{10!} 2\pi \, \mathrm{i} \; v(K, 0) = -\frac{2\pi i}{10!}(-1) = \frac{2\pi i}{10!}$$