Needham, Chapter 8, Exercise 1, p. 420
G. Palmer, June 1, 2017, 5:30 pm PST

1 Thinking of x as representing time, z(x) = x + i f(x) is a parametric description of the ordinary graph y = f(x).

(i) Show that the complex velocity is v = 1 + i tan θ, where θ is the angle between the horizontal and the tangent to the graph.

$$v = \frac{dz}{dx} = 1 + i\frac{df}{dx} = 1 + i\,\frac{dy}{dx} = 1 + i\tan\theta$$

Also, show that the complex acceleration is a = i f''.

$$\frac{dv}{dx} = i\,\frac{d^2y}{dx^2} = i\,f''$$

(ii) Recall from Ex. 20 on page 262 that the curvature of the orbit is $\kappa = [\mathrm{Im}\,(a\,\overline{v})] / |\,v\,|^3]$. Deduce from (i) that

$$\kappa \sec^3\theta = f''(x)$$

Answer:

$$|\,v\,|^2 = (1 + i\tan\theta)(1 - i\tan\theta) = 1 + \tan^2\theta = \sec^2\theta$$

$$\kappa = [\mathrm{Im}(a\,\overline{v})] / |\,v\,|^3]$$

$$\kappa\ |\,v\,|^3 = \mathrm{Im}(i\,f''\,(1 - i\tan\theta)) = \mathrm{Im}(i\,f'' + f''\tan\theta)$$

$$\kappa \sec^3\theta = f''$$

(iii) From (ii), deduce that the error equation (3) can be written as

$$\text{area}(ABCD) = \tfrac{1}{8} f''(x)\,\Delta^3. \qquad (1)$$

The error equation (3) calculates an infinitesimal error in a complex curve.

$$\text{area}(ABCD) = \left(\tfrac{1}{8}\kappa \sec^3\theta\right)\Delta^3 \qquad (3)$$

Since $\kappa \sec^3\theta = f''$ from (ii), the result (1) is obtained with a single substitution.