Needham, Chapter 7, Notes
G. Palmer, Oct 24, 2017, 4:10 pm PDT

Notes on notation in Chapter VII, Winding Numbers and Topology

This chapter introduces the notation for winding functions. I found much of it difficult to follow. I'm not sure why, but I think the notation has something to do with it. There are at least five flavors of L, including $\hat{L}$, which appears to be equivalent to or at least tightly wrapped onto the unit circle, and there are flavors of z and w that correspond to L and $\hat{L}$. L and $\hat{L}$ are used for objects; $\mathcal{L}$'s, variously marked are used for transformations. There are transformations from loops to unit circles, from unit circles to loops, and from unit circles to loops *on* unit circles. Both L and Γ are loops. The difference that would warrant two symbols for a loop is not clear. Vasco wrote in the Forum that L implies a limitation to real functions of a single variable and Γ implies loops in the complex plane.

This is not a complete list of the notation used in Chapter 7. It is just a list of notation in the problematic sections.

General Notation

C, a unit circle

L, a loop in any shape (p. 342)

$\mathcal{L}$(C), deformation of C into another loop (p. 342); L = $\mathcal{L}$(C) (inference)

$\mathcal{L}$(z), transformation of z (a point in C) into w (a point in L) (p. 342)

$\hat{L}$, "standardized loop on unit circle" (p. 342) consisting of points $\hat{w}$; $\hat{L} = \mathcal{L}_s(\text{L})$, where s = 1 (inference)

$\hat{\mathcal{L}}(e^{i\theta})$, a point $\hat{w}$; a mapping from C to $\hat{L}$; $\hat{\mathcal{L}}(e^{i\theta}) = e^{i\Phi(\theta)}$ (p. 343)

$\mathcal{L}_s$(z), a transformation of a point w in L to a point radially closer to a point on $\hat{L}$; $\mathcal{L}_s(z) = w + s(\hat{w} - w)$ (p. 343) [Should this be $\mathcal{L}_s$(w)?] $\mathcal{L}_s$(L) would be a deformation of L (inference).

$\Phi(\theta)$, the positive rotational angle between w and $\mathcal{L}(e^{i0})$ (p. 342)

θ, angle from 0 to 2π (p. 342)

w, a point in a loop L [5]; $\mathcal{L}(e^{i\theta})$, $R(\theta)\, e^{i\Phi(\theta)}$

$\hat{w}$, a point on unit circle with same Φ as w; $\hat{w} = \hat{\mathcal{L}}(e^{i\theta}) = e^{i\Phi(\theta)}$ (p. 343)

$\hat{\mathcal{J}}_\nu$, "archetypal mapping of degree ν"; $\hat{\mathcal{J}}_\nu(z) = z^\nu$, "for which $\Phi(\theta) = \nu\theta$ (p. 343)

Γ, a circle (not necessarily a unit circle) or a simple loop in general (p. 350, [10]); why Γ rather than L?

Winding Numbers and Multiplicity

a, typically a p-point or a root

p-point, the pre-image of a mapping to a point p (p. 345)

root, preimage of 0; solution to f(z) = 0 (p. 345)

n, an integer value of algebraic multiplicity or winding number; power of Δ = (z-a) in Taylor's series

winding number and degree, *ν*

ν(a), the winding number around an infinitesimal circle centered at a; the topological multiplicity of a; the preimage of a zero point p of a function; sign of det[J(a)]

ν[L, 0] (p. 338) "net number of revolutions of the direction of z as it traces out L once in its given sense"

ν[L, p] (p. 339), *nu*, the topological winding number of a loop L about a point p; obtained by traversing loop by inspection or by counting intersections with ray; degree of mapping $\hat{\mathcal{L}}$; "loop $\hat{L}$ of winding number *ν*" (p. 343); "winding number *ν*, (p. 344)

degree, "In this context [windings] it is common to speak of degree of mapping $\hat{\mathcal{L}}$ which produces $\hat{L}$, rather than of the 'winding number' of L (or L)"; "The degree of $\hat{L}$ (i.e. *ν*)"; "the archtypal mapping of degree *ν* in $\hat{\mathcal{J}}_n$(z)" (p. 343)

winding number and polynomials

ν[P(Γ:)], *If a **simple** loop Γ winds once around m roots of P(z), then ν[P(Γ), 0] = m* (p. 345, boldface added)

winding number and p-points

N, If "N is the number of p-points [counted with their multiplicities] inside Γ, then N = *ν*[f(Γ), p]"

algebraic multiplicity and analytic functions

algebraic multiplicity, "root a is a zero point of algebraic multiplicity n" (p. 346); multiple copies of root "a"; algebraic multiplicity of an analytic f can be represented as a convergent Taylor series in the neighborhood of a nonsingular point (p. 346); found by finding power n of Δ = (z-a) in

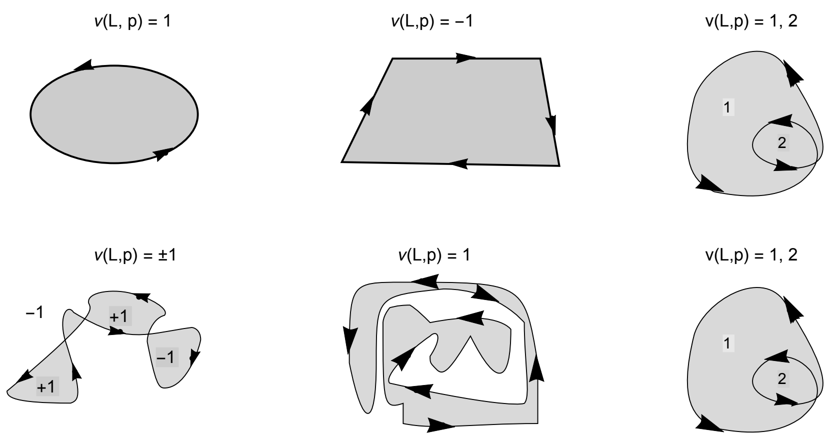
$\frac{f(a)^{(n)}(a)}{n!}\Delta^n$ (p. 347); "we define the algebraic multiplicity of a to be n" (p. 347); algebraic multiplicity n is $\nu[h(\Gamma_a)],\ p]$ (348); v(a) = n, i.e. v(a) = algebraic multiplicity (p. 348);

algebraic multiplicity and continuous mappings

algebraic multiplicity, "If the algebraic multiplicity is n, then the winding numbers of the image will also be n" (p. 347); " 'multiplicity' for a mapping h(z) that is merely continuous" (p. 348)

topological multiplicity ν(a) and continuous mappings

topological multiplicity, ν(a), "define the topological multiplicity of a to be $\nu(a) = \nu[h(\Gamma_a),\ p]$ (p. 348); notice that topological multiplicity is the multiplicity of a p-point and it is defined by the winding number of its image; "*local degree* of h at a" (p. 348, footnote 3); the number of p-points inside Γ counted with their topological multiplicities equals ν[h(Γ), p] (p. 350, (10));



p. 339, paragraph 1, *It is often useful to consider the winding number of a loop about a point p other than the origin, and this is correspondingly written ν(L, p). Instead of counting the revolutions of z, we now count those of (z-p). For example, the shaded region in [1] can be defined as all the positions of p for which ν(L, p) ≠ 0. Try shading this set for the other loops.* [See above figure after Figure [1], p. 338].

p. 343, ¶ 1, line 2. *If z moves at unit speed round C, what does the slope (including the sign) of the graph represent*?

[UNFINISHED]

Chapter 7, Notes and suggested exercises, p. 346.

p. 346, paragraph 5, If the root a of a polynomial P(z) has multiplicity n then P may be factorized as $A^n \Omega(z)$*, where* $\Omega(a) \neq 0$*. It follows by simple calculation [exercise] that the first (n-1) derivatives of P vanish at a, so that a is a critical point of order at least (n-1).*

Define $(z-a)^0$ = (z-a)' = 1, where 0 is the power of A in the first summand of $P^{(n)}$.

$$\text{P} = A^n\,\Omega(z) = (z-a)^n\Omega(\text{z})$$

$$P^{(1)} = \text{n}(z-a)^{n-1}\Omega(\text{z}) + (z-a)^n\Omega'(\text{z})$$

In the series $P^{(1)}$ .. $P^{(n)}$, for each $P^{(m)}$, m in (1..n), the $A^{n-m} = (z-a)^{n-m}$ in the first summand has the lowest power of A. Terms with powers ≥ 1 vanish at a. All P in $\{P^{(1)}$ .. $P^{(n-1)}\}$ consist entirely of terms that vanish at a. Since $P^{(n-1)}$ contains $(z-a)^1 = 0$ in the first summand, all derivatives from (1) up to (n-1) vanish at a. Since $P^{(n-1)} = 0$, a is a critical point of order (n-1) (p. 204). The nth derivative contains $(z-a)^0 = 1$, which does not vanish at a. Then at a,

$$P^{(n)} = n!\,(z-a)^0\Omega(\text{z}) = \text{n!(z-a)'}\Omega(\text{z}) = \text{n!}\Omega(\text{z})$$

which is non-zero.

Notes on Chapter 7, IV.3, p. 349. What's topologically special about analytic functions?

ν(a) for critical and non-critical points in analytic and nonanalytic functions

Analytic functions

non-critical points: +1
critical points: positive integer other than 1, but not zero

Nonanalytic functions

non-critical points: ±1
critical points: 0, ±1, other integer, positive or negative

Notes from the pizza oven.

p. 353, ¶ 1 *By following the effect of the transformation on little loops round each of the two preimages, we confirm this prediction: one preimage has multiplicity + 1 while the other has multiplicity -1.*

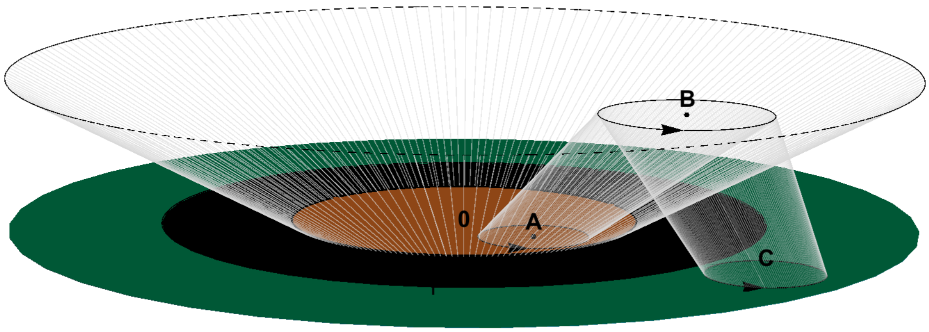


Figure 1: Rotation and preimages on merged layers

The outer rim of the black ring has radius 1. The brown inner disc and the black ring together constitute the original pizza dough. When the pizza has been lifted and stretched out, it is pressed down flat, resulting in two layers over the disc labeled "C", which lies outside the unit circle. Since there is no pizza within the outer green layer itself, the overlying layers are from the top layer with the B disc and from the slanted sidewall arising from lifting up the inner disc and stretching a thin ring of the pastry lying outside of the circle surrounding the inner disc. It appears from Figure 1 that the small circles around A and B must rotate in the same direction. The preimage encircling A has multiplicity +1. Then according to the text on p. 353, there must be a small image circle in the layer from the sidewall that rotates in the opposite direction. This is not evident from the plot. It might be that one has to view the rotation from outside the wall (or from below the C disc). This image circle would also have a preimage conguent with the circle around A but with opposite orientation. But what is the *geometric* rationale for saying that the preimages have opposite orientations and multiplic ities?

Notes on Rouché's Theorem and Brouwer's Fixed Point Theorem (BFPT), exercises 12 (p. 372) and 15 (pp. 373-374).

Rouché's Theorem and BFPT are needed to understand Exercise 12, which is prerequisite to Exercise 15. The four topics are grouped together here.

p. 353, Rouché's Theorem (RT)

The essentials:

$|g| < |f(z)|$ on a simple loop Γ (not $\leq$)
$\nu[(f+g)(\Gamma), 0] = \nu[f(\Gamma), 0]$
(The winding number of (f+g) equals the winding number of f.)
(f+g) has the same number of roots ("zeroes") inside Γ as f.

p. 354, Brouwer's Fixed Point Theorem (BFPT), a "slightly different result"

This is not a proof of BFPT, but rather a "slightly different result" in which $|g(z)| < 1$ rather than $|g(z)| \leq 1$. We switch from a simple loop Γ to a disc D such that $|z| \leq 1$ on D. But we set a condition: $|g(z)| < 1$ for all z in D. Hence, g(z) is in D, but not on the boundary. Then, we let m(z) = g(z) - z, and we let f(z) = - z. This assures that $g(z) + f(z) \neq 0$ on the boundary (unit circle), because $|f(z)| = 1$ and $|g(z)| < 1$. Then, by Rouché's Theorem, m(z) has the same number of roots in D as f(z). The number of roots is 1, because f(z) = -z and $\nu[f(C), 0] = 1$. Notice that for RT, we considered the number of roots inside a simple loop Γ and for BFPT we consider the number of roots in D, which is the disc including its boundary. The construction in the proof of the narrow result satisfies the conditions of RT.

p. 355, line 4: "On the boundary of D ..." can be more readily understood when expanded to "For z on the boundary of D ..."

p. 372, Exercise 12, a "slight" generalization of Rouché's Theorem.

(i) If p(z) and q(z) are not 0 on Γ,

$$\tilde{\Gamma} = p(z)\, q(z)(\Gamma)$$

(ii) $$\nu[\tilde{\Gamma}, 0] = \nu[p(\Gamma), 0] + \nu[q(\Gamma), 0] \qquad (1)$$

$$f(g) + g(z) = f(z)\left[1 + \frac{g(z)}{f(z)}\right] = f(z)\, H(z)$$

As in Rouché's Theorem, the condition is established that $|g(z)| < |f(z)|$ on Γ. We can see that H is anchored at 1, and since $\left|\frac{g(z)}{f(z)}\right| < 1$, we can see that $H(z) > 0$ for all z in Γ, so H(z) cannot wind around zero and $\nu[H(z), 0] = 0$. Substituting f(z) for p(z) and H(z) for q(z) in (1), we obtain Rouché's Theorem.

(iii) relaxes the condition on the lengths of g(z) and f(z) so that

$$|g(z)| \leq |f(z)|$$

and suggests that "$\nu[H(\Gamma), 0]$ might not be well-defined." We are to show that if we stipulate $f + g \neq 0$ on Γ, then $\nu[H(\Gamma), 0] = 0$," as in (ii). This looks suspiciously like what we did in (ii). Of course, if the winding number is not well-defined, it cannot be subtracted or added.

pp. 373-374, Exercise 15, Generalization of BFPT

Rouché's Theorem was generalized so it could be used to prove a more general version of BFPT. For a proof by contradiction, we assume that there is no fixed point in some continuous mapping of "the disc itself", "so that m ≠ 0 throughout the disc |z| ≤ 1."

(i) As in the discussion on p. 354, "m(z) = g(z) - z = g(z) + f(z) does not vanish **on the unit circle C**" [emphasis added]. By Ex. 12 (iii) we have shown that ν[m(C), 0] = 1. But if m(z) were to vanish at any point or every point on Γ, then it could not make a full wind, so ν[m(C), 0] = 0. It is important not to confuse ν[H(z), 0] = 0 with ν[m(C), 0] = 0. We want ν[H(z), 0] = 0 so that ν[m(C), 0] = 1. Ex. 12 (iii) established that ν[H(z), 0] = 0 where |g(z)| ≤ |f(z)| and perhaps even where m(z) vanishes, i.e. f + g = 0. Why is this problem developed on C, when the BFPT applies to the whole disc? Perhaps that question is answered in (ii), which I do not understand.

(ii) appears to require that we show that $\nu[m(C_r), 0] = 0$ at r = 1, because that would create a contradiction with (i). In my posted answer, I showed that the winding was undefined at r = 0, but it seemed to be 1 where 0 < r <= 1. In either case, I don't see a contradiction, and I don't see how this would support the proof by contradiction.

End of Notes on Rouché's Theorem and Brouwer's Fixed Point Theorem (BFPT), exercises 12 (p. 372) and 15 (pp. 373-374).

[Exercise], Chapter 7, Section VII, 1 Schwartz's Lemma, p. 359, paragraph 1

Show that f = $M_c \circ M_0^{\phi}$.

It is given that f is an analytic function with a well-defined inverse that maps the disc to itself, but does not leave the centre fixed, rather it moves it to c. A rigid motion M_c sends c back to 0. The composition $M_c \circ$ f maps the disc to itself, but leaves the centre fixed. By Schwartz's lemma, there is a rotation $M_0^{\phi} = M_c \circ$ f.

We take M_c to be an instance of the involutory Möbius transformation $M_a = \frac{z-a}{\bar{a}\,z-1}$, which maps a disc to itself and swaps 0 to a and a to 0 (p. 179). Then

$$M_0^{\phi} = M_c \circ f$$

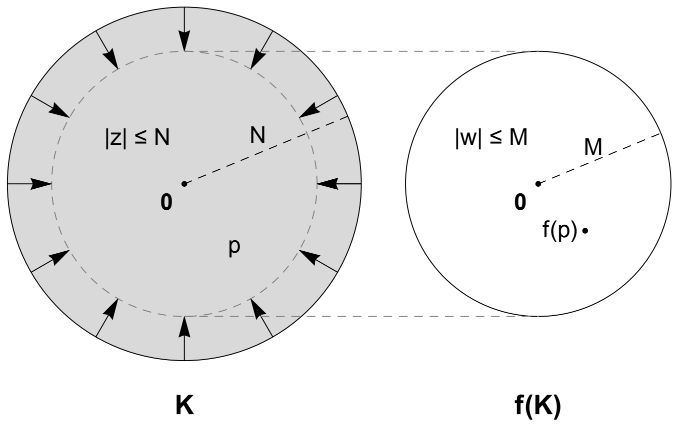
$$M_c^{-1} \circ M_0^{\phi} = M_c^{-1} \circ M_c \circ f$$

$$M_c \circ M_0^{\phi} = f, \qquad \text{because } M_c^{-1} = M_c$$

This is what we wanted.

pp. 359-360, §§ 2, Liouville's Theorem

There are no suggested exercises in this subsection, but when I reread it in connection with Exercise 2, Chapter 9, it struck me as a tricky section. I found it helpful to make a figure.



$$|F(p)| = \left|\frac{f(p)}{p}\right| \leq \max[F(z)] \text{ on } K$$

Schwarz: Every interior point moves closer to center (or it is a rotation) (p. 358).

Then f(p) must be closer to center than p, and

$$|F(p)| \leq \max[F(z) \text{ on } K] = [\ \frac{\max|f(z)|}{N} \text{ on } K\ ] \leq \frac{M}{N},$$

because max|f(z)| occurs on the boundary, K.

[Exercise] VIII, 2 Poles and essential singularities, p. 366

The order of a pole is the order of the first nonvanishing derivative of (1/f).

P(z) = 1/sin(z)

P(z) has a pole wherever sin(z) is 0, which happens whenever $z = n\pi$, n = 0,1,2,..∞. Needham pointed out that "P has a simple pole at each multiple of π".

F = 1/P(z) = sin(z) vanishes at 0
F' = cos(z)

Examine the value of F' at the poles.

$$F'(n\pi) = \cos(n\pi) = \{1, -1\}$$

The first derivative at $z = n\pi$ is nonzero. Hence, the order of every one of the infinite poles is 1.

Q(z) = cos $z \big/ z^2$

Q(z) has a double pole at z = 0.

F = 1/Q(z) = z^2/cos z

F' = $\frac{2\,z\cos z - z^2\,(-\sin z)}{(\cos z)^2} = \frac{2\,z\cos z + z^2\sin z}{\cos^2 z}$ vanishes at 0

$F^{(2)} = 2\sec(z) + z^2\sec^3(z) + 4\,z\sec(z)\tan(z) + z^2\sec(z)\tan^2(z)$

$F^{(2)}(0) = 2$

The first nonvanishing derivative of F has order 2, so the double pole of Q has order 2.

R(z) = $\frac{1}{(e^z-1)^3}$

R(z) has a pole whenever $e^z = 1$. This happens when z = 0, so R(z) has a triple pole at zero. If z is treated as an angle iθ, then z = 0 every n2π for n = 0,1,2,...∞, and the number of triple poles is infinite.

F = $(e^z - 1)^3$

F' = $3\,e^z\,(e^z - 1)^2$ vanishes at z = 0

$F^{(2)} = 6\,e^{2\,z}\,(e^z - 1) + 3\,e^z\,(e^z - 1)^2$ vanishes at z = 0

$F^{(3)} = 6\,e^{3\,z} + 18\,e^{2\,z}(e^z - 1) + 3\,e^z\,(e^z - 1)^2$

$F^{(3)}(0) = 6$

The first nonvanishing derivative of F has order 3, so each of the infinite triples of F has order 3.

p. 366, ¶ 4

g(z) = $e^{1/z}$

|g(z)| = $e^{\frac{\cos\theta}{r}}$

I missed this problem somewhere along the path of complex number calculations, so I was puzzled for a bit how to get from g(z) = $e^{1/z}$, z = r$e^{i\theta}$ to the absolute quantity. Just add an ounce of Euler.

Rewrite

$$g(z) = e^{1/z} = e^{\frac{1}{re^{i\theta}}} = e^{\frac{e^{-i\theta}}{r}} = e^{\frac{\overline{\cos\theta + i\sin\theta}}{r}} = e^{\frac{\cos\theta - i\sin\theta}{r}} = e^{\frac{\cos\theta}{r}} e^{\frac{-i\sin\theta}{r}}$$

Then

$$|g(z)| = e^{\frac{\cos\theta}{r}} \left|e^{\frac{-i\sin\theta}{r}}\right| = e^{\frac{\cos\theta}{r}}$$