Needham, Chapter 7, Exercise 9, p. 372.
G. Palmer, April 18, 2017, 4 pm PST

9 Referring to the previous exercise, consider the differential equation

$$\frac{d^3Q}{dt^3} - Q = 0$$

(i) Find $\mathcal{R}$ for this equation. Does it satisfy the Nyquist Stability Criterion?

(ii) Confirm your conclusion by explicitly solving the differential equation.

$\mathcal{R}$ is the net rotation of F(s) as s traverses the imaginary axis from bottom to top (by Ex. 8).

Substitute e^{st} for Q to obtain a specific differential equation:

$$\frac{d^3e^{st}}{dt^3} - e^{st} = 0$$

$$s^3 e^{st} - e^{st} = 0$$

$$F(s) = s^3 - 1 = 0$$

e^{st} is never 0, so there are three roots: the cube roots of unity: $\{1, e^{i2\pi/3}, e^{-i2\pi/3}\}$ (See pp. 26-27). As s traverses the imaginary axis from bottom to top, $\mathcal{R}$ is 2π, not $n\pi = 3\pi$, because one of the roots (the root 1) has Re(s) > 0. The equation does not satisfy the Nyquist Stability Condition and it will not decay away with time.

(ii) solving the differential equation.

In Exercise 8, Needham wrote: *One solves this equation by taking a linear superposition of special solutions of the form $Q_j(t)$*
$= e^{s_j t}$. Before proceeding further, rewrite the pair of conjugate roots in a tidier format:

$$e^{i2Pi/3} = \cos(2\pi/3) + i \sin(2\pi/3)$$

$$= -1/2 + i\sqrt{3}/2$$

$$e^{-i2Pi/3} = -1/2 - i\sqrt{3}/2$$

Then

$$Q = A\, e^{s_1 t} + Be^{s_2 t} + Ce^{s_3 t}$$

$$= A\, e^{t} + B\, e^{\left(-1/2+i\sqrt{3}/2\right)t} + C\, e^{\left(-1/2-i\sqrt{3}/2\right)t}$$

And note that the second and third terms having Re(s) < 0 die out with increasing t, but the first term expands with increasing t.

Note also:

$$\left(-1/2 + i\sqrt{3}/2\right)^3 = 1$$

$$(-1/2 - i\sqrt{3}/2)^3 = 1$$

We test some solutions:

s = 1, A = 1

$$\frac{d^3 e^t}{dt^3} - e^t = 0$$

$$e^t - e^t = 0$$

s = 1 is a solution.

$s = e^{i2\pi/3}$, B = 1

$$\frac{d^3 e^{e^{i2\pi/3}\,t}}{dt^3} - e^{e^{i2\pi/3}\,t} = 0$$

$$\left(e^{i2\pi/3}\right)^3 e^{e^{i2\pi/3}\,t} - e^{e^{i2\pi/3}\,t} = 0$$

$$e^{e^{i2\pi/3}\,t} - e^{e^{i2\pi/3}\,t} = 0$$

$s = e^{i2\pi/3}$ is a solution.

$s = e^{-i2\pi/3}$, C = 1

$$\frac{d^3 e^{e^{-i2\pi/3}\,t}}{dt^3} - e^{e^{-i2\pi/3}\,t} = 0$$

$$\left(e^{-i2\pi/3}\right)^3 e^{e^{-i2\pi/3}\,t} - e^{e^{-i2\pi/3}\,t} = 0$$

$$e^{e^{-i2\pi/3}\,t} - e^{e^{-i2\pi/3}\,t} = 0$$

$s = e^{-i2\pi/3}$ is a solution.

We test the general solution:

$$\frac{d^3\left(A\,e^t + B\,e^{\left(-1/2 + i\sqrt{3}/2\right)t} + C\,e^{\left(-1/2 - i\sqrt{3}/2\right)t}\right)}{dt^3} - \left(A\,e^t + B\,e^{\left(-1/2 + i\sqrt{3}/2\right)t} + C\,e^{\left(-1/2 - i\sqrt{3}/2\right)t}\right) = 0$$

$$= Ae^t + B\left(-1/2 + i\sqrt{3}/2\right)^3 e^{\left(-1/2 + i\sqrt{3}/2\right)t} + C\left(-1/2 - i\sqrt{3}/2\right)^3 e^{\left(-1/2 - i\sqrt{3}/2\right)t} - \left(A\,e^t + B\,e^{\left(-1/2 + i\sqrt{3}/2\right)t} + C\,e^{\left(-1/2 - i\sqrt{3}/2\right)t}\right)$$

$$= Ae^t + B\,e^{\left(-1/2 + i\sqrt{3}/2\right)t} + C\,e^{\left(-1/2 - i\sqrt{3}/2\right)t} - \left(A\,e^t + B\,e^{\left(-1/2 + i\sqrt{3}/2\right)t} + C\,e^{\left(-1/2 - i\sqrt{3}/2\right)t}\right)$$

$$= 0$$

The linear superposition is a solution.