Needham, Chapter 7, Exercise 7, p. 371.
G. Palmer, April 12, 2017, 3:18 pm PST

7 Consider the nonanalytic mapping $h(z) = |z|^2 - i\overline{z}$.

(i) Find the roots of h.

$|z|^2 - i\overline{z}$

$z\,\overline{z} - i\,\overline{z} = 0$

$\overline{z}\,(z-i) = 0$, $z\,(\overline{z} + i) = 0$

The roots (zeroes) of h are {i, 0}.

(ii) Calculate the Jacobian J, and hence find det(J).

Let z = a = x + iy.
Rewrite $h(z) = (x^2 + y^2)$ -i(x-iy) = $(x^2 + y^2)$ -y -ix.

$u = x^2 + y^2 - y$
$v = -x$

$$J = \begin{pmatrix} 2x & 2y - 1 \\ -1 & 0 \end{pmatrix}$$

det(J) = 2y-1

Note that when y = 1/2, z is a critical point, so the whole line y = 1/2 consists of critical points. This may have implications for (vi).

(iii) Use (9) to calculate the multiplicities of the roots in (i). (9) says that topological multiplicity $\nu(a)$ *= the sign of det[J(a)].*

y(a)	ν(a)
0	-1
1/2	---
>1/2	+1
<1/2	-1

Then $\nu(0) = -1$ and $\nu(\mathrm{Im}(i)) = \nu(1) = +1$.

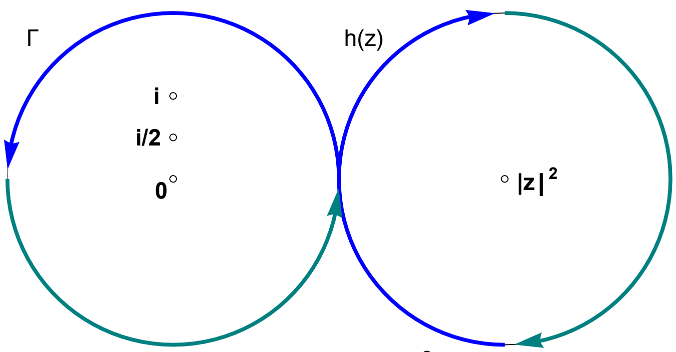

Figure 1: $h(z) = |z|^2 - iz$

(iv) Find the image curve traced by h(z) as $z = 2\,e^{i\theta}$ *traverses the circle |z| = 2, and confirm the prediction of the Topological Argument Principle.*

The Topological Argument Principle:

The total number of p-points inside Γ (counted with their topological multiplicities) is equal to the winding number of h(Γ) round p. (p. 350)

Vasco's answer has a good analysis of this. Write $h(z) = z\bar{z} - i\bar{z}$. Let $z = 2\,e^{i\theta}$ and let $\bar{z} = 2\,e^{-i\theta}$. Then as z traverses Γ counter clockwise, $\bar{z}$ is traveling clockwise and the two rotations $\bar{z}z$ cancel each other out. The p-points are the two roots of h. The product $\bar{z}z = 2\,e^{i\theta}2\,e^{-i\theta} = 4$ has arg θ-θ = 0.

The second product ($-i\bar{z}$) rotates $\bar{z}$ by $-\pi/2$. As z makes a traversal, $\bar{z}$ makes one loop in the clockwise direction, beginning at -Pi/2 and ending at -5Pi/2. This loop is added to and centred at 4. It is evidently only this loop that shows up in Figure 1, h(z).

Thus, we see that there are actually three loops. The first two have opposite orientation. The third is out of phase with the first two. There is still agreement with the TAP because the topological multiplicity associated with the roots sums to zero, while we see in h(z) that none of the loops wind around the origin. Hence, One could say that $|\,z\,|^2$ in h(z) removes all the loops beyond the possibility of encircling the origin. This confirms that [ν(0) + ν(i)] = *ν*[h(z), 0].

(v) Gain a better understanding of the above facts by observing that h(z) = $\bar{z}$ *(z-i) and then mimicking the analysis of [8].*

By the analysis of [8], while z traverses Γ about 0 and i undergoing a rotation of 2π, $\bar{z}$ undergoes a rotation of -2π about 0 and i. We are considering no points outside of Γ defined as (|z| = 2). Then the loops induced by z and $\bar{z}$ cancel. In any case, neither loop encircles 0.

(vi) Use the insight of the previous part to find *ν*(i/2), which cannot be done with (9).

ν(i/2) is undefined by (iii), as the problem statement suggests. In (ii) it was shown that the line y = 1/2 consists entirely of critical points. By inspection of Γ, we see that i/2 lies within Γ between 0 and i on the y axis, so this point might also be describable by *ν*(a) = 0. Following Vasco's answer, we consider a small circle C centred at z = i/2. C does not encircle z = i (or 0). As z goes round C once, z goes round once in the opposite direction and -iz goes round once in the opposite direction and out of phase. None of the image loops in h(z) encircle 0, so using the same argument as in (iv) we find by the continuity of h(z) that *v*(i/2) = 0 = *v*[h(C), 0] or perhaps we could write *v*(i/2) = -1 = [h(C), h(i/2)].