Needham, Chapter 7, Exercise 6, p. 370.
G. Palmer, April 11, 2017, 1:20 pm PST

6 Even in three or more dimensions the local linear transformation induced by a mapping f at a point a can still be representated by the Jacobian matrix J(a), and if a is not a critical point then its topological multiplicity $\nu(a)$ as a preimage of f(a) is still given by (9). If n is the number of real negative eigenvalues of J(a), counted with their algebraic multiplicities, show that

$$\nu(a) = (-1)^n$$

[Hint: Since the characteristic equation det[J(a)-λ I] = 0 has real coefficients, any complex eigenvalues must occur in conjugate pairs.]

Let ξ, η, μ, etc. be infinitesimal orthogonal vectors anchored at a non-critical point *a*. We can write the characteristic equation (determinant) of the Jacobian of the local linear transformation induced by the mapping f at *a* as:

$$(\xi - \lambda_1)(\eta - \lambda_2)(\mu - \lambda_3) \ldots = 0$$

$$\lambda_1\lambda_2\lambda_3 \ldots + A\lambda_1\lambda_2 \ldots + B\lambda_2\lambda_3 \ldots + C\lambda_1\lambda_3 \ldots = 0$$

where A, B, and C ... are expressions in ξ, η, μ, ...

Since complex eigenvalues always come in conjugate pairs, they will always have a positive product.

The sign of the product of positive real eigenvalues is always positive.

The sign of the product of negative real eigenvalues is given by $(-1)^n$.

The first term dominates the CE and therefore the sign of the CE. Hence, the sign of the determinant of f equals $(-1)^n$.