Needham, Chapter 7, Exercise 4, p. 370.
G. Palmer, April 13, 2017, 9:20 pm PST

5 As in the text, let ξ_a, η_a, ϕ_a denote the two perpendicular expansion factors and the rotation angle used to describe the local linear transformation at a produced by a mapping. By considering the case of a rotation by (π/4), for which J is constant, show that ξ and η are generally not the eigenvalues λ_1 and λ_2 of the Jacobian J. However, confirm for this example that det(J) = $\lambda_1 \lambda_2 = \xi\eta$.

Rotation matrix for π/4:

$$[R^{\pi/4}] = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The expansion factors can be written as columns of a 2 x 2 matrix.

$$[M] = \begin{pmatrix} \xi_a & 0 \\ 0 & \eta_a \end{pmatrix}$$

Write the Jacobian.

$$J = [R^{\pi/4}].[M] = \begin{pmatrix} \frac{\xi_a}{\sqrt{2}} & \frac{-\eta_a}{\sqrt{2}} \\ \frac{\xi_a}{\sqrt{2}} & \frac{\eta_a}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_a & -\eta_a \\ \xi_a & \eta_a \end{pmatrix} \qquad (1)$$

Write the characteristic equation.

$$\det[J - \lambda I] = 0$$

$$\det\left[\begin{pmatrix} \frac{\sqrt{2}\,\xi_a}{2} & -\frac{\sqrt{2}\,\eta_a}{2} \\ \frac{\sqrt{2}\,\xi_a}{2} & \frac{\sqrt{2}\,\eta_a}{2} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\left(\frac{\sqrt{2}\,\xi_a}{2} - \lambda\right)\left(\frac{\sqrt{2}\,\eta_a}{2} - \lambda\right) + \frac{\xi_a\,\eta_a}{2} = 0$$

$$\lambda^2 - \frac{\sqrt{2}\,\xi_a}{2}\lambda - \frac{\sqrt{2}\,\eta_a}{2}\lambda + \xi_a\,\eta_a = 0$$

In order for the roots to be equal to ξ_a and η_a, the characteristic equation would have to factor into $\left(\frac{\sqrt{2}\,\xi_a}{2} - \lambda\right)\left(\frac{\sqrt{2}\,\eta_a}{2} - \lambda\right)$. This CE has the extra term $\frac{\xi_a\,\eta_a}{2}$. Therefore, *ξ and η are generally not the eigenvalues λ_1 and λ_2 of the Jacobian J*.

By Viete's formulas <en.wikipedia.org/wiki/Quadratic_equation>, the product of the roots of the quadratic is c/a.

$$\lambda_1 \lambda_2 = c/a = \xi_a \eta_a$$

This is what we wanted, but it is not clear that this is a determinant. We are to confirm for this example that *det(J)* = $\lambda_1 \lambda_2 = \xi\eta$.

$$\det\left[\begin{pmatrix} \frac{\sqrt{2}\,\xi_a}{2} & -\frac{\sqrt{2}\,\eta_a}{2} \\ \frac{\sqrt{2}\,\xi_a}{2} & \frac{\sqrt{2}\,\eta_a}{2} \end{pmatrix}\right] = \xi_a \eta_a$$

So now we have det[J] = $\lambda_1 \lambda_2 = \xi_a \eta_a$, which is what we wanted.

Note that det[J] is not the same as the characteristic equation. Also note that we could have treated [M] as the Jacobian. The determinant of [M] represents the area of a parallelogram with sides ξ_a and η_a. Rotation of the vector matrix by $\pi/4$ does not change the length of the sides or the angle between the vectors. Therefore, the area of the parallelogram and the determinant should remain constant. The determinant of [M] is $\xi_a \eta_a$ and its characteristic equation is $(\xi_a-\lambda)(\eta_a-\lambda) = 0$, which gives $c/a = \lambda_1 \lambda_2 = \xi_a \eta_a$.

It makes one wonder why rotating [M] produces a different characteristic equation. Is there a contradiction here? Should the rotation be omitted from the calculation of the characteristic equation and the eigenvalues?