Needham, Chapter 7, Exercise 4, Part v, p. 370.
G. Palmer, April 17, 2020, 10 am PST

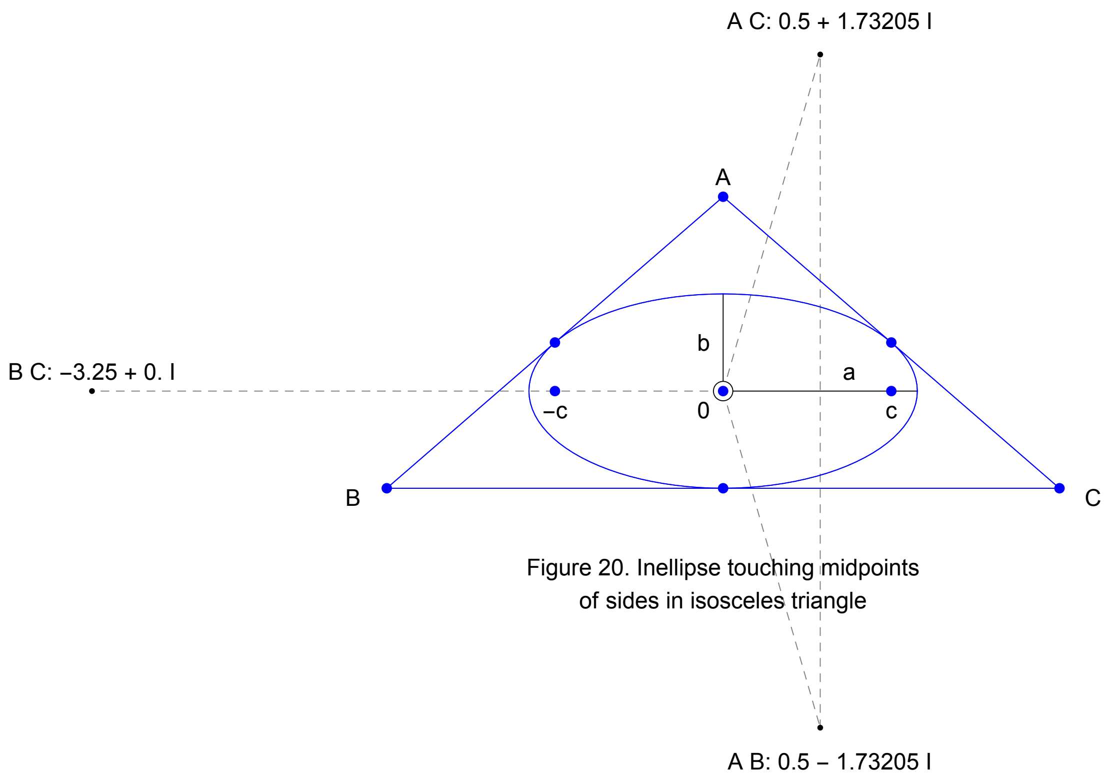

Figure 20. Inellipse touching midpoints of sides in isosceles triangle

(v) [Hard] Show that the two critical points of f are the foci of $\mathcal{E}$!

Consideration of ellipses generally involves some consideration of semi-minor and -major axes, which are generally given the symbols a and b, and consideration of focal points, which are often given the symbol c. These can be easily confused with the names for the vertices given by Needham. To avoid that particular confusion, after introducing essential equations, we rename the vertices.

From the vantage point of the ellipse, the foci expressed as distance from the centroid are

$$c = \pm\sqrt{a^2 - b^2}\ ,$$

where a and b are the lengths of major and minor axes, respectively.

From the vantage point of the polygon, the critical points are

$$z = \frac{(a+b+c)\pm\sqrt{(a+b+c)^2 - 3\,(ab+ac+bc)}}{3},$$

where a, b, and c are the complex roots of the polygon P = (z-a)(z-b)(z-c). Since we would like to make an equation that includes both foci and roots of the polygon, we rename the roots of the polygon to A, B, and C (which is easier to type than subscripted forms of z or ξ). Then we have

$$z = \frac{(A+B+C)\pm\sqrt{(A+B+C)^2 - 3\,(A\,B + A\,C + B\,C)}}{3}$$

If we knew that c and z were both on the real axis, then we could write c = z and

$$c = \sqrt{a^2 - b^2} = \frac{(A+B+C)\pm\sqrt{(A+B+C)^2 - 3\,(A\,B + A\,C + B\,C)}}{3} = z$$

which relates the foci to the critical points. This is what we wish to show. We know that an inelipse touching an isosceles triangle at the midpoints has foci and centre on the major axis, which is parallel to the real axis. If we plot the enclosing triangle using the vertices as the roots, the base of the triangle is parallel to the real axis. Since we know that this ellipse can be transformed to an arbitrary inellipse touching the midpoints of the sides of a similarly transformed triangle, there is no loss of generality from using an isosceles trangle for the inellipse. Moving either the base of the triangle or the major axis of the ellipse to the x axis involves no loss of generality. We choose to put the major axis of the ellipse on the x axis and the centre of the ellipse and centroid of the triangle at the origin. The centroid of the ellipse and the triangle are the same point 0 = A+B+C. Therefore, we can simplify the equation for the foci in terms of the critical points.

$$c = \frac{\sqrt{-3\,(A\,B + A\,C + B\,C)}}{3}$$

We use only the positive root because we have defined c to be the positive focus. In Carlson's proof of Marden's theorem (telemeter, Forum), we find a relation between the polynomial $P = \Pi(z - z_i)$ and the focus:

$$p = \Sigma z_i\, z_{i+1} = -3\,c^2$$

where p is a factor in the coefficient of z in the reduced form of the polynomial $P(z) = z^3 + pz + q$ (For another form, see also, Needham, p. 4, bottom). We will take this as given and write

$$c = \sqrt{\frac{-p}{3}} = \frac{\sqrt{-3\,p}}{3}$$

This is the key item in the proof. We can rewrite $\Sigma_{z_i\, z_{i+1}}$ as (AB + AC + BC). Then

$$p = AB + AC + BC$$

$$c = \frac{\sqrt{-3\,p}}{3} = \frac{\sqrt{-3\,(A\,B + A\,C + B\,C)}}{3}$$

Showing that c is the critical point. Since Carlson's proof also has $c = \sqrt{a^2 - b^2}$, it is proven by Carlson and shown here that the two critical points of f are the foci of $\mathcal{E}$.

Observation on the relation between foci and critical points with inellipse touching midpoints

of sides of isosceles triangle

Remarkably, AC = $\overline{\text{AB}}$, so AC + AB is real and BC is real. That is, the multiples of the two vertices of each of the two sides of an isosceles triangle with centroid at the origin and base parallel to the real line are conjugate (Figure 20), so their sum is real. The multiple of the two vertices of the base is real. Then

$$\text{c} = \frac{\pm\sqrt{-3\,(\text{Re}\,(\text{AB+AC})\text{+BC})}}{3}$$

or

$$c^2 = -\,(\text{Re}(\text{AB + AC})\text{+BC})/3$$