Needham, Chapter 7, Exercise 4, Part ii, p. 370.
G. Palmer, April 9, 2020, 2:40 pm PST

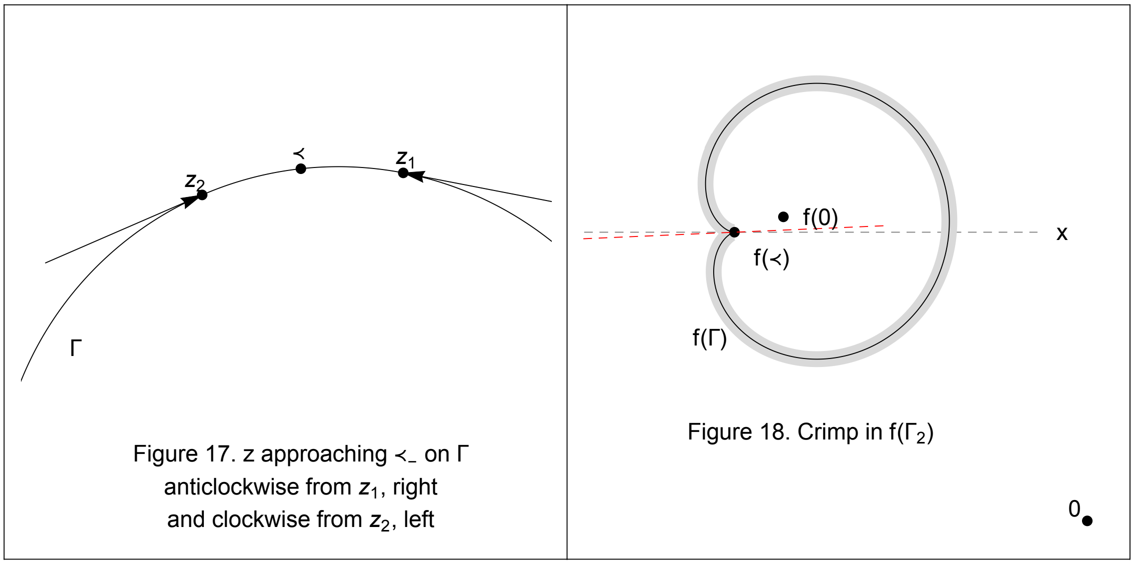

Figure 17. z approaching ≺_ on Γ anticlockwise from z_1, right and clockwise from z_2, left

Figure 18. Crimp in $f(\Gamma_2)$

(ii) If f'(p) ≠ 0 then a little piece of Γ passing through p is merely amplitwisted to another almost straight piece of curve through f(p). Deduce that ≺ shapes can only occur when Γ hits a critical point. Explain why the particular shape is consistent with a critical point of order 1.

The crimp can only occur when Γ hits a critical point because f(z) is conformal everywhere but at critical points. At non-critical points, angles are preserved.

Explain why the particular shape is consistent with a critical point of order 1.

Figure 17 shows a plot of z at points z_1 and z_2 as z approaches a critical point ≺ in Γ_2 from opposite directions. In Figure 18, the black curve is the image of Γ under f(z). The underlying gray shadow is the image of Γ under the Taylor series for f(z). The black and gray curves have identical shape because the Taylor series for a polynomial is just the polynomial itself.*

First determine that the critical points are of order 1. The section that seems most likely to provide an answer is section IV, subsection 1, pp. 346-347. But one cannot immediately adapt what it says there. First, the function in ℂ is not as straightforward as that in Figure [9], where f(z) is real valued. Second, the shape shown in [9c] is not precisely the crimp shape that we have with the complex cubic. Third, the discussion in paragraph 4 does not seem to work for a polynomial with all terms of multiplicity 1 (because m = 1, 1-1 = 0).

So we have to go to the factorization $A^n\,\Omega(z)$ to find the multiplicity of a root, as Needham explained on p. 347. The only arguments to f(z) are points of Γ, which can be written as $re^{i\theta}$, where r = | ≺ |, where < is a critical point. As the absolute values of the two critical points of a cubic polynomial are generally not equal, one can plot a Γ for each critical point. We have to consider "the rarer case in which a is a critical point" (para. 1, p. 347) and $\Omega(z) = \frac{f^{(2)}(\prec)}{2!} + \frac{f^{(3)}(\prec)}{3!}\Delta$, with $f^{(2)}$

being the first nonvanishing derivative of P; and $a = \prec$. For the cubic, the derivatives higher than $f^{(3)}$ do not exist. Then we can write the equation

$$f(z) - p = \Omega(z)\,\Delta^2$$

$$= \left(\frac{f^{(2)}(\prec)}{2!} + \frac{f^{(3)}(\prec)}{3!}\,\Delta\right)\Delta^2 = \frac{f^{(2)}(\prec)}{2!}\Delta^2 + \frac{f^{(3)}(\prec)}{3!}\Delta^3$$

$$= \frac{f^{(2)}(\prec)}{2!}(z-\prec)^2 + \frac{f^{(3)}(\prec)}{3!}(z-\prec)^3$$

Since the first non-vanishing derivative contains $\Delta^2 = (z-\prec)^2$, we set m = 2 and the order of the critical point is (m – 1) = 1 (See p. 205). Why the first nonvanishing derivative rather than the second? The magnitude of the Taylor series depends most on the term with Δ^2, which dominates because Δ^3 is tinier for sufficiently small Δ.

Then if p = f(a), we can add p to both sides and substitute $\prec$ for a to get

$$f(z) = f(\prec) + \frac{f^{(2)}(\prec)}{2!}(z-\prec)^2 + \frac{f^{(3)}(\prec)}{3!}(z-\prec)^3 \qquad (1)$$

which can be used to plot the gray curve in figure 18.

Now return to paragraph 3 of the section and figure [9] illustrating the idea that a crimp is formed when a second point b on Γ approaches a critical point until b arrives, at which point the slope vanishes. We know the first derivative vanishes at the critical points. But in $\mathbb{C}$, it appears that rather than decreasing the slope at a critical point as in the real function, the approach of the second point causes the **difference** in slopes on either side of $f(\prec)$ to shrink. Or one could say that as z_1 and z_2 approach $\prec$ on Γ from opposite sides, the image slopes on either side of $f(\prec)$ are reflected about a central axis passing through the critical point at some unknown angle. If we allow z to approach the critical point (or depart from it) from both sides at equal distances along Γ_2 from a critical point, the curve near $\prec$ appears as a crimp symmetrical about its own axis. This is because $|\Delta| = |z-a|^n$ becomes smaller on both sides equally, that is by an amount approximately proportional to the square of the shrinking distance of z from $\prec$. This is why the crimp shape is consistent with a critical point of order 1.

Vasco has explained why $|\Delta|$ shrinks equally on both sides of $\prec$. In my attempt to illustrate his answer, I have used some different notation, so my reproduction might seem quite different, but I think it is essentially the same. We write $\prec = re^{i\phi}$ and $z = re^{i(\phi \pm \psi)}$, where ψ is the angular distance of z from $\prec$. $(-\psi)$ is increasing negative as z_1 rotates away from $\prec$, and $(+\psi)$ is increasing positive as z_2 rotates away from $\prec$. For both $\prec$ and z, $r = |\prec|$. Equation (1) can be rearranged and rewritten

$$f(z) - f(\prec) = \frac{f^{(2)}(\prec)}{2!}(z-\prec)^2 + \frac{f^{(3)}(\prec)}{3!}(z-\prec)^3 \qquad (2)$$

Assume that z starts at the critical point and rotate z away from the point in both directions. We write $\delta|f(z) - f(\prec)|$ to indicate the change in $|f(z) - f(\prec)|$. We wish to show that for every increment in ψ,

$$\delta\,|\,f(z_2) - \mathrm{f}(\prec)| = \delta\,|\,f(z_1) - \mathrm{f}(\prec)|$$

Substituting for z and $\prec$ gives

$$(z_1 - \prec) = \mathrm{r}\,(e^{i(\phi-\psi)} - e^{\mathrm{i}\phi})$$
$$(z_2 - \prec) = \mathrm{r}\,(e^{i(\phi+\psi)} - e^{\mathrm{i}\phi})$$

Then, by Ex. 18, Ch. 1,

$$\mathrm{r}\,(e^{i(\phi-\psi)} - e^{\mathrm{i}\phi}) = 2\ \mathrm{r}\ \mathrm{i}\ \sin\left[\frac{-\psi}{2}\right] e^{\frac{i(2\phi-\psi)}{2}}$$
$$\mathrm{r}\,(e^{i(\phi+\psi)} - e^{\mathrm{i}\phi}) = 2\ \mathrm{r}\ \mathrm{i}\ \sin\left[\frac{\psi}{2}\right] e^{\frac{i(2\phi+\psi)}{2}}$$

$$(z - \prec)^2 = -4\,r^2\,\sin^2[\tfrac{\pm\psi}{2}]e^{\mathrm{i}2\phi}\,e^{\pm\psi} \qquad (3)$$

Then, if we look just at the dominant term with the 2nd derivative in the RHS of (2),

$$\mathrm{f(z)\ -\ f}(\prec) = \frac{f^{(2)}(\prec)}{2!}\left(-4\,r^2\,\sin^2\left[\frac{\pm\psi}{2}\right] e^{i(2\,\phi\pm\psi)}\right) \qquad (4)$$

The angle of the sides of the crimp and therefore of the axis of the crimp is determined by the RHS of (4), and particularly by $f^{(2)}(\prec)e^{i(2\,\phi\pm\psi)}$. Arg($f^{(2)}(\prec)$) is determined by the placement of the roots, so it must be calculated or treated as an unknown constant. The term $e^{i(2\,\phi\pm\psi)}$ can be written $e^{\mathrm{i}2\phi}\,e^{\pm\psi}$, showing that the point $f^{(2)}(\prec)e^{\mathrm{i}2\phi}$ is rotated equally in opposite directions. Then the point should be an axis of rotation, but this is not the whole story. Using just the dominant 2nd derivative works well for the magnitude of the vector near the critical point, but for the angle we need to include the 3rd derivative. Calculating arg(f(z) – f($\prec$)) as (arg($f^{(2)}(\prec) + f^{(3)}(\prec)$)), one obtains an axis that appears to bifurcate the crimp (Figure 18, red dashed line).

It remains to explain the crimp shape itself. For this we need the absolute values of ($f(z_1)$ – f($\prec$) and ($f(z_2)$ – f($\prec$)).

$$|\mathrm{f(z)\ -\ f}(\prec)| = |\frac{f^{(2)}(\prec)}{2!}||(z - \prec)^2| = |\frac{f^{(2)}(\prec)}{2!}|\ |\text{-}4\ r^2\ \sin^2[\tfrac{\pm\psi}{2}]e^{i(2\,\phi\pm\psi)}|$$

In the context of a critical point of a polynomial with fixed roots, $|\frac{f^{(2)}(\prec)}{2!}|$ and $\left|-4\,r^2\right|$ are constants. They can be combined into a single constant A; and $|e^{i(2\,\phi\pm\psi)}| = 1$.

$$|\mathrm{f(z)\ -\ f}(\prec)| = \mathrm{A}\ \sin^2[\tfrac{\pm\psi}{2}]$$

As z_1 and z_2 rotate equally away from $\prec$, the distance of f(z) from f($\prec$) amplifies equally by an amount proportional to the square of the sine of the angular distance from the critical point. This is what produces the crimp shape. We have shown that a crimp shape symmetrical about a particular axis determined by the derivatives of the Taylor series is consistent with a critical point of order 1.

An example

Where the roots are labeled a, b, c and

$$f(z) = (z-a)(z-b)(z-c) = z^3 - (a + b + c)\, z^2 \ + \ (a\, b + b\, c + a\, c)\, z - a\, b\, c$$

The first derivative $f^{(1)}$ has degree 2.

$$f^{(1)}(z) = 3\, z^2 - 2\ (\mathrm{a + b + c})\mathrm{z} + \ (\mathrm{a\, b + b\, c + a\, c})$$

Because we seek the roots, we set this equal to 0 and solve for z to obtain the critical points $\prec_-$ and $\prec_+$. These are points which produce zero in the first term of the Taylor's series as on p. 347 (or the second term as in (1) above).

$$\mathrm{z} = \frac{(\mathrm{a+b+c}) \pm \sqrt{\ (\mathrm{a+b+c})^2 - 3\ (\mathrm{ab + ac + bc})}}{3} = \prec_\pm$$

Since the critical points are based on the assumption that $f^{(1)}(\prec_\pm) = 0$, we know that the $f^{(1)}$ term of the Taylor's series vanishes. The value of the equation at $\prec_\pm$ now depends the $f^{(2)}$ and $f^{(3)}$ terms.

$$f^{(2)} = 6\mathrm{z} - 2(\mathrm{a+b+c})$$

$$f^{(3)} = 6$$

Substituting these derivatives into (1) for f(z) using the Taylor's series

$$\mathrm{f(z) = f(}\prec_-) + (3 \prec_- - (\mathrm{a+b+c}))\ (z - \prec_-)^2 + (z - \prec_-)^3$$

On p. 205, Needham wrote *The behavior of a mapping very near to a critical point is essentially given by z^m, $m \geq 2$. Rather than being conserved, angles at the critical point are essentially multiplied by m. We quantify the degree of this strange behavior by saying that z = 0 is a critical point of order (m-1).* We choose quadratic exponent of the first derivative as m. Then the critical points have order (m-1) = 1.

Notes, Part (ii)

*The Taylor series of a polynomial is just the polynomial itself (https://en.wikipedia.org/wiki/Taylor_series). Vasco (p.c., Forum) has given a nice explanation of this using a cubic polynomial:

If we write the Taylor series for the cubic f about the point q then it has a finite number, 4, of terms *because we only have 3 derivatives:*

$$f(z) = f(q) + f'(q)(z - q) + (1/2!)f''(q)(z - q)^2 + (1/3!)f'''(q)(z - q)^3$$

Since we know that the LHS is a polynomial of degree 3 by definition, and we can see that the RHS is also a polynomial of degree 3, then if they are equal the RHS must just be another way of writing the same polynomial. It's clear that this is true for any polynomial.

The identity of the polynomial and the Taylor series is illustrated in Figure 18 with the black curve f(z) superimposed on the gray curve plotted with the Taylor series.

(iii) Observe that there are only two points in the evolution of Γ at which a $\prec$ shape is produced. Explain this algebraically in terms of the degree of f'.

The two (critical) points at which the $\prec$ shape is produced are shown as $\prec_-$ and $\prec_+$ in Figure 1. f' is quadratic, so it has degree 2 and two roots obtained by the quadratic equation. The roots of the quadratic are the critical points, i.e. the zero points of f'.

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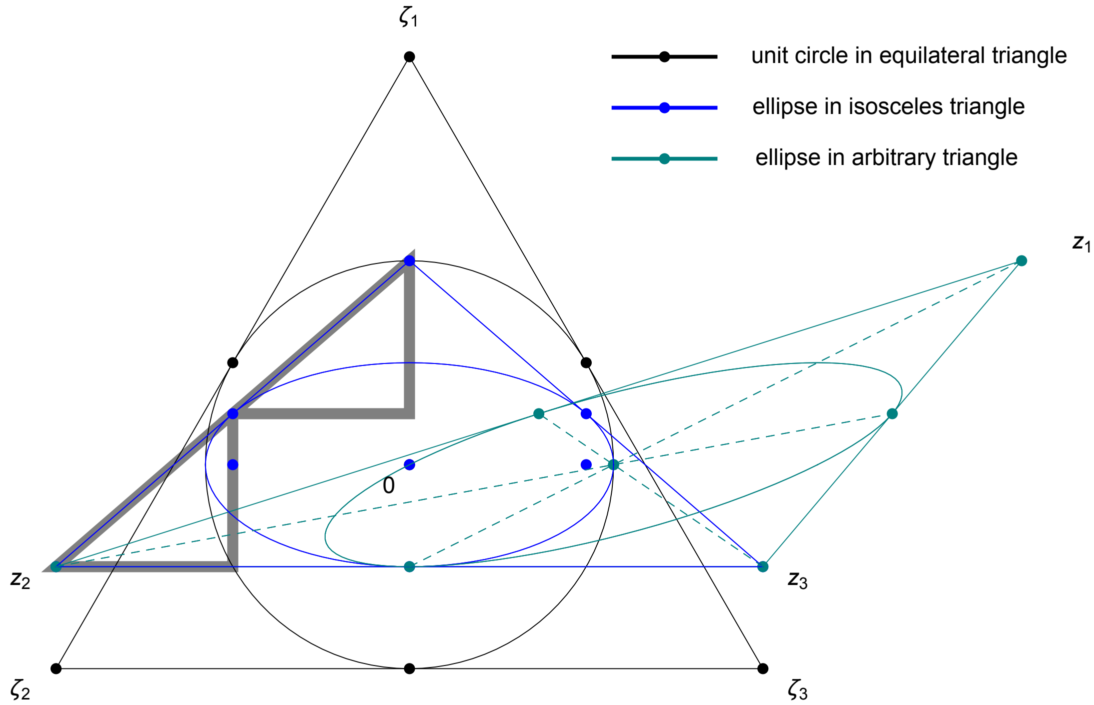

Figure 19. Compression and shear
on unit circle in equilateral triangle

(iv) Let T be the triangle with vertices a, b, and c. There are many ellipses which can be inscribed in T so as to touch all three sides, but show that there is only one (call it $\mathcal{E}$) that touches T at the midpoints of the sides.

Figure 19 shows an equilateral triangle with the unit circle inscribed (black). The blue figure represents a second plane in which the y-axis is compressed by (b/a), where a and b are the major and minor axes of the blue ellipse ($\mathcal{E}_1$). Notice that the tangent points of the triangle on the ellipse are still at the midpoints of the triangle sides, but the triangle has become isoseles. The green figure represents a third plane created by applying shear to the blue plane parallel to the x-axis and above the line $z_2\, z_3$. The tangent points of the triangle on the green ellipse ($\mathcal{E}$) show that the inscribed curve conserves tangent points at the midpoints of the triangles through two linear transformations of the plane.

A circle on three predetermined points is unique. An ellipse can be seen as a linear transformation of a circle. An ellipse can be produced by stretching or contracting a circle along an axis. These are linear transformations. Therefore, if a circle (or “incircle”) touching three sides of a triangle at the midpoints is transformed on one or both of two orthogonal axes by stretching or contracting until it becomes an interior ellipse (or “inellipse”) still touching the three midpoints, the ellipse must be unique. If the base of the original triangle is not parallel to the x-axis, it can be rotated to parallel prior to transformation and translated to the x-axis if desired, and then rotated and translated back to its original orientation and position without loss of generality. Rotation about its center or about the origin has no effect on the shape of the circle, which is symmetric in every direction.

There appear to be three basic possibilities for modifying the shape of the circle or ellipse to obtain a new ellipse: We can amplify (or compress) it, subject it to shear, or rotate it. Shear combines amplification and rotation.

(1) Amplification in the form of compression is applied to $\mathbb{C}$ with z ⟶ Re(z) + I (b/a) Im(z) and illustrated by the blue figure, preserves the contact at the midpoints because it preserves similar triangles with corresponding sides equal as seen in the thick gray triangles on the left side of the blue figure. That the two triangles are indeed congruent, is demonstrated by the fact that the lower triangle was plotted by simple translation of the upper triangle. The compression has no effect on the base, so the midpoint touchpoint in the base remains unchanged. If compression were to be applied to the circle independently of the triangle, the resulting ellipse would lose tangency with two of the midpoints. The transformation is linear, so it preserves parallel lines and midpoints (p. 208), or more generally proportions on lines, but it is not analytic, because it fails the Cauchy-Riemann test (p. 210).

(2) Application of shear to the plane of the blue triangle and ellipse produces a result illustrated with the green triangle and ellipse. This result could be accomplished by translating the base of the blue triangle to the x axis, applying the shear, and translating it back to its original position. After the first translation, shear is applied with z ⟶ (scale Im[z] + z), where scale is a real number determining the amount of shear parallel to the x-axis. Again, the transformation is linear, so it preserves parallel lines and midpoints, but it is not analytic by the Cauchy-Riemann test. The effect on the sides $z_1\,z_2$ and $z_1\,z_3$ and on the ellipse is linear amplification, so the touchpoints on the ellipse remain at the midpoints of the triangle, as shown by the dashed lines passing through the centroid of the green triangle. Again, there is no effect on the midpoint of the base. There is rotation about the origin by approximately $d\theta = \sqrt{dr^2 - dx^2}/r$, which can be rewritten more precisely by the cosine rule as

$$\theta = \cos^{-1}\left(\left(|z_0|^2 + |z_1|^2 - (\mathrm{Re}(z_1) - \mathrm{Re}(z_0))^2\right) \big/ (2\,|z_0\,z_1|)\right).$$ [These z's are not the vertices shown in Figure 19.]

If shear were to be applied to the ellipse independently of the triangle, the resulting new ellipse would not maintain tangency at the midpoints. One can see from Figure 19, that the northwest side of $\mathcal{E}$ would shear away from side $z_1\,z_2$ and the east end of $\mathcal{E}$ would break through the side $z_1\,z_3$ leaving intersections rather than a tangent at the mid-point. One might try and correct with positive rotation of the ellipse about it's own center, but the shear moves the centroid of the ellipse eastward of the centroid of the triangle, meaning that the ellipse can no longer touch all three sides at the midpoints. An intriangle touching the midpoints of the outer triangle loses contact when its centroid is moved.

(3) Rotation of the ellipse and the triangle about the origin equally would change neither the shapes nor their tangencies. Rotation of the ellipse independently of its enclosing triangle would create a different ellipse and loss of tangency.

We conclude that for a given triangle (such as z_1, z_2, z_3), there is only one ellipse (the green one; call it $\mathcal{E}$) that touches T at the midpoints of the sides.