**Needham, Chapter 7, Exercise 4, p. 370.**
**G. Palmer, March 26, 2020, 10:20 am PST**

Out[3621]=

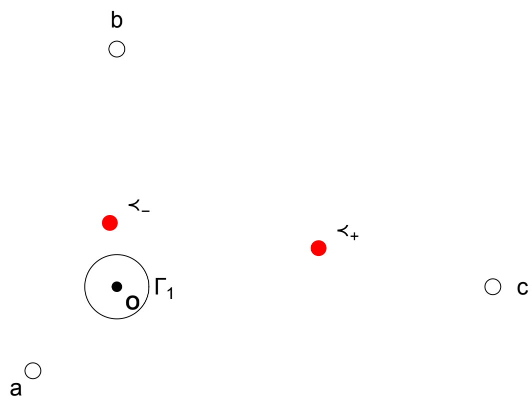


Figure 1. Inner circle $\Gamma_1$, critical points $\prec_-$ and $\prec_+$ of cubic $f(z) = (z-a)(z-b)(z-c)$

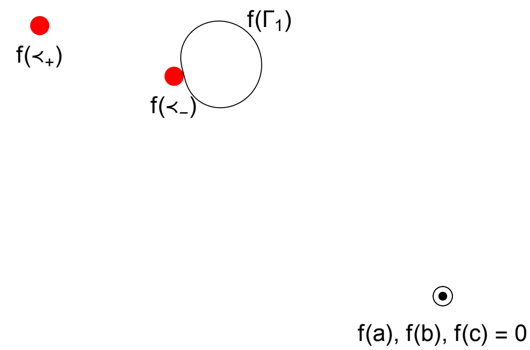


Figure 2. $f(\Gamma_1)$, $f(\prec_-)$ and $f(\prec_+)$ of cubic $f(z) = (z-a)(z-b)(z-c)$ and $f(a)$, $f(b)$, $f(c)$

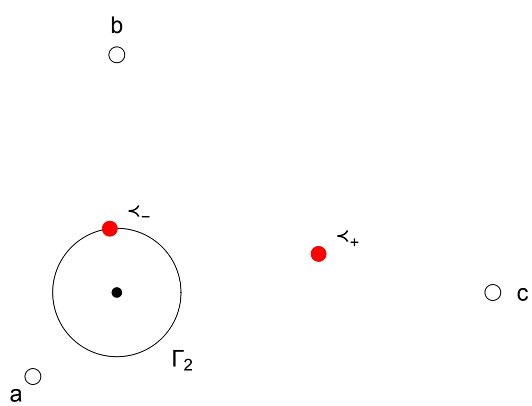


Figure 3. $\Gamma_2$ passes through $\prec_-$

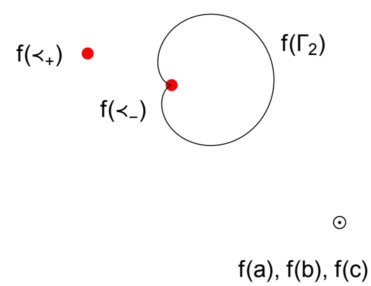


Figure 4. $f(\Gamma_2)$ has crimp

Out[3655]=

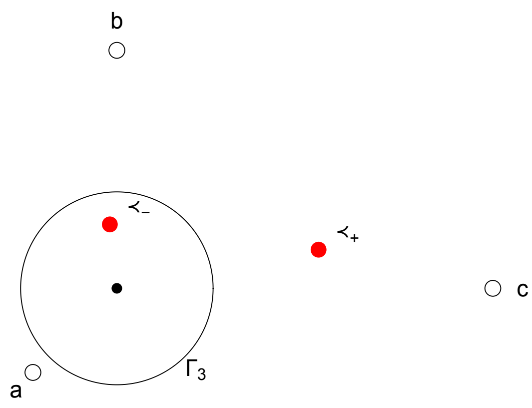


Figure 5. $\Gamma_3$ between $\prec_-$ and $\prec_+$

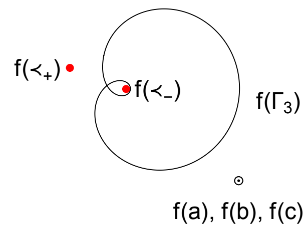


Figure 6. $f(\Gamma_3)$ has inner loop encircling $f(\prec_-)$

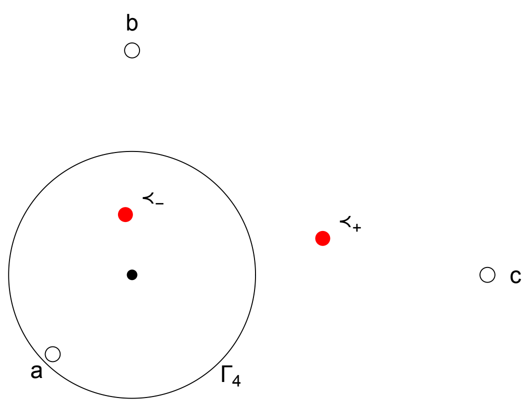


Figure 7. $\Gamma_4$ outside $\prec_-$ and a

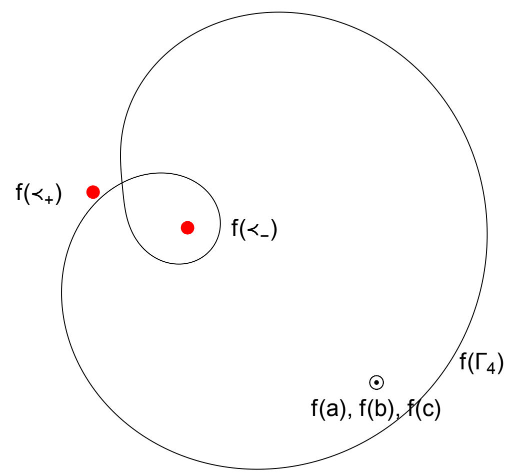


Figure 8. $f(\Gamma_4)$ enlarges to encircle f(a)

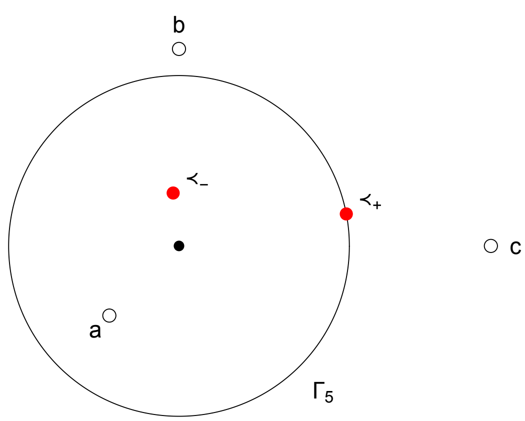


Figure 9. $\Gamma_5$ crosses $\prec_+$

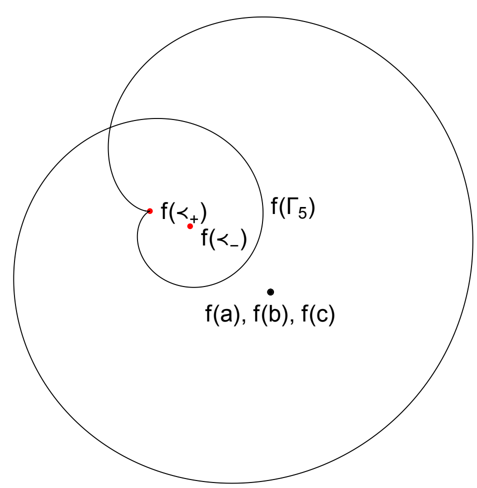


Figure 10. Crimp appears in $f(\Gamma_5)$

Out[3689]=

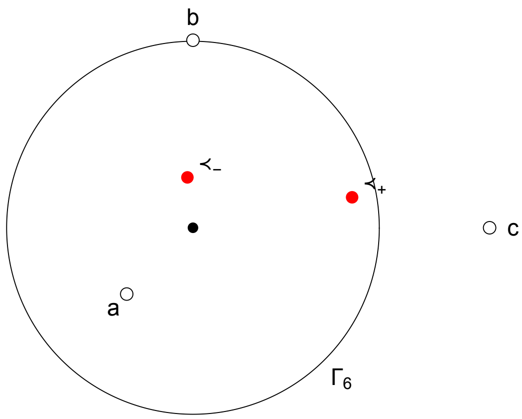


Figure 11. $\Gamma_6$ crosses $\prec_+$

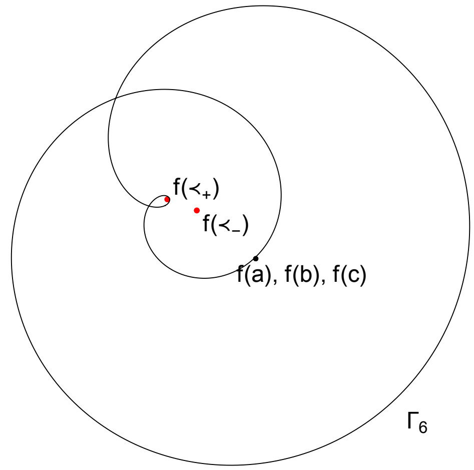


Figure 12. New loop appears in $f(\Gamma_6)$

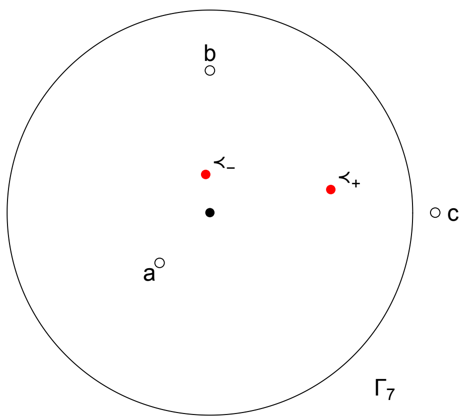


Figure 13. $\Gamma_7$ encircles a and b

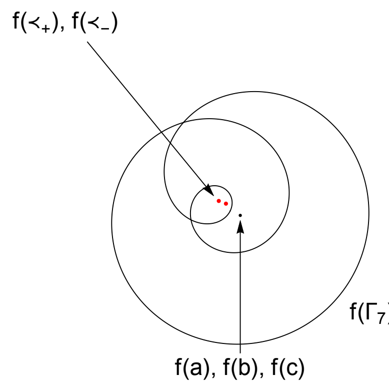


Figure 14. Loops of $f(\Gamma_7)$ encircle images of a, b, c

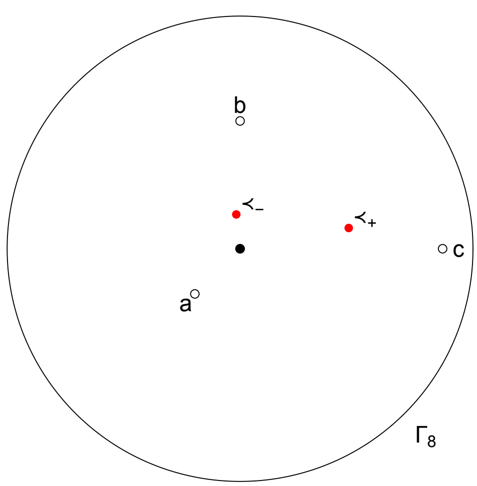


Figure 15. $\Gamma_8$ encircles a, b, and c

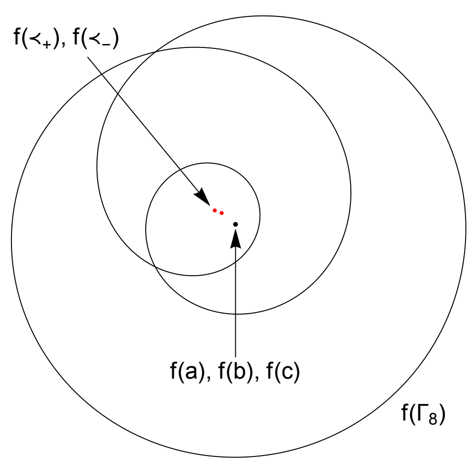


Figure 16. Loops of $f(\Gamma_8)$ encircle images of a, b, c

*4 Reconsider [8].*

*(i) Use a computer to draw the image under a cubic mapping*

$$f(z) = (z-a)(z-b)(z-c)$$

*of expanding circle Γ, and observe the manner in which the winding number increases as Γ passes through the roots a, b, and c. In particular, observe that the shape that marks the birth of a new loop is this: "≺".*

This symbol for the crimp will be used for the critical points indexed as $\{ \prec_+, \prec_- \}$ by the sign of the square root in the solution of the quadratic, as shown in part (ii).

The left column of Figures 1-14 shows the pre-image circles, the roots of f(z), and the critical points $\prec_+$ and $\prec_-$. The calculation of the critical points is described in part (ii). The right column shows the image-loops and image-points. Notice that the crimp shapes form precisely at the images of critical points. f(z) gains a loop which provides an additional "wind" for every image of a critical point or root that it encircles. We count the winds about the images of critical points separately from winds about the images of roots.

**Figures 1, 2**

$\Gamma_1$ encircles no critical points. In Figure 2, one can see that $f(\Gamma_1)$ is distorted where $\Gamma_1$ is in the neighborhood of the nearest critical point $\prec_-$.

$$\nu[f(\Gamma_1), f(\prec_-)] = \nu[f(\Gamma_1), f(\prec_+)] = 0 \qquad \text{[See p. 345].}$$

For the single image of the roots, a, b, c, we can write:

$$\nu[f(\Gamma_1), 0] = 0$$

**Figures 3, 4**

As the radius of Γ grows, Γ passes through the critical point $\prec_-$ and $f(\Gamma_2)$ shows a crimp at the point. Since $f(\Gamma_2)$ does not encircle $f(\prec_-)$ this will not be counted as a wind.

$$\nu[f(\Gamma_2), f(\prec_-)] = \nu[f(\Gamma_2), f(\prec_+)] = 0, \quad \nu[f(\Gamma_2), 0] = 0$$

**Figures 5, 6**

The radius of $\Gamma_3$ extends beyond $\prec_-$, so that $\prec_-$ is encircled by $\Gamma_3$. The image $f(\Gamma_3)$ buds a new inner loop which encircles $f(\prec_-)$.

$$\nu[f(\Gamma_3), f(\prec_-)] = 2, \quad \nu[f(\Gamma_3), f(\prec_+)] = 0, \quad \nu[f(\Gamma_3), 0] = 0$$

**Figures 7, 8**

The radius of $\Gamma_4$ extends beyond the root a but encircles no new critical points. The image $f(\Gamma_4)$ enlarges to encircle f(a) = 0.

$$\nu[f(\Gamma_4), f(\prec_-)] = 2, \quad \nu[f(\Gamma_4), \mathrm{f}(\prec_+)] = 0, \quad \nu[f(\Gamma_4), 0] = 1$$

**Figures 9, 10**

As $\Gamma_5$ crosses $\prec_+$, a crimp forms in the image on the inner loop at $\mathrm{f}(\prec_+)$.

$$\nu[f(\Gamma_5), \mathrm{f}(\prec_-)] = 2, \quad \nu[f(\Gamma_5), f(\prec_+)] = 1, \quad \nu[f(\Gamma_5), 0] = 1$$

**Figures 11, 12**

$\Gamma_6$ encircles both $\prec_-$ and $\prec_+$. A new loop encircling $\mathrm{f}(\prec_+)$ develops within the inner loop. It appears that as a curve $\Gamma$ expands to encircle new critical points, each critical point $\prec_\pm$ more distant from the origin results in a new loop in $\mathrm{f}(\Gamma)$ that forms within the previous innermost loop and encircles $\mathrm{f}(\prec_\pm)$. The winding number $\nu(\mathrm{f}(\prec_+))$ leapfrogs to 3 over $\nu(\mathrm{f}(\prec_-))$.

$$\nu[f(\Gamma_6), \mathrm{f}(\prec_-)] = 2, \quad \nu[f(\Gamma_6), \mathrm{f}(\prec_+)] = 3, \quad \nu[f(\Gamma_6), 0] = 1$$

**Figures 13, 14**

$\Gamma_7$ is allowed to encircle both critical points and two roots. The result is that $\mathrm{f}(\Gamma_7)$ encircles the image of all three roots within the first inner loop and both critical points within the second, innermost loop. See also, [8b], p. 345.

$$\nu[f(\Gamma_7), f(\prec_-)] = 3, \quad \nu[f(\Gamma_7), f(\prec_+)] = 3, \quad \nu[f(\Gamma_7), 0] = 2$$

**Figures 15, 16**

$\Gamma_7$ is allowed to encircle both critical points and all three roots. The result is that $\mathrm{f}(\Gamma_7)$ encircles the image of all three roots and both critical points within the innermost loop. See also, [8b], p. 345.

$$\nu[f(\Gamma_7), f(\prec_-)] = 3, \quad \nu[f(\Gamma_7), f(\prec_+)] = 3, \quad \nu[f(\Gamma_7), 0] = 3$$

Needham cites the Argument Principle on p. 345:

*If f(z) is analytic inside and on a simple loop Γ, and N is the number of p-points [counted with their multiplicities] inside Γ, then N = ν[f(z), p].*

Here p stands for a single point in the f(z) plane. In this exercise, the point 0 has three preimages (p-points), so when $\Gamma$ encircles all three, then $\mathrm{f}(\Gamma)$ encircles the point 0 three times and N=3.