Needham, Chapter 7, Exercise 3, p. 370.
G. Palmer, March 28, 2017, 9:30 pm PST

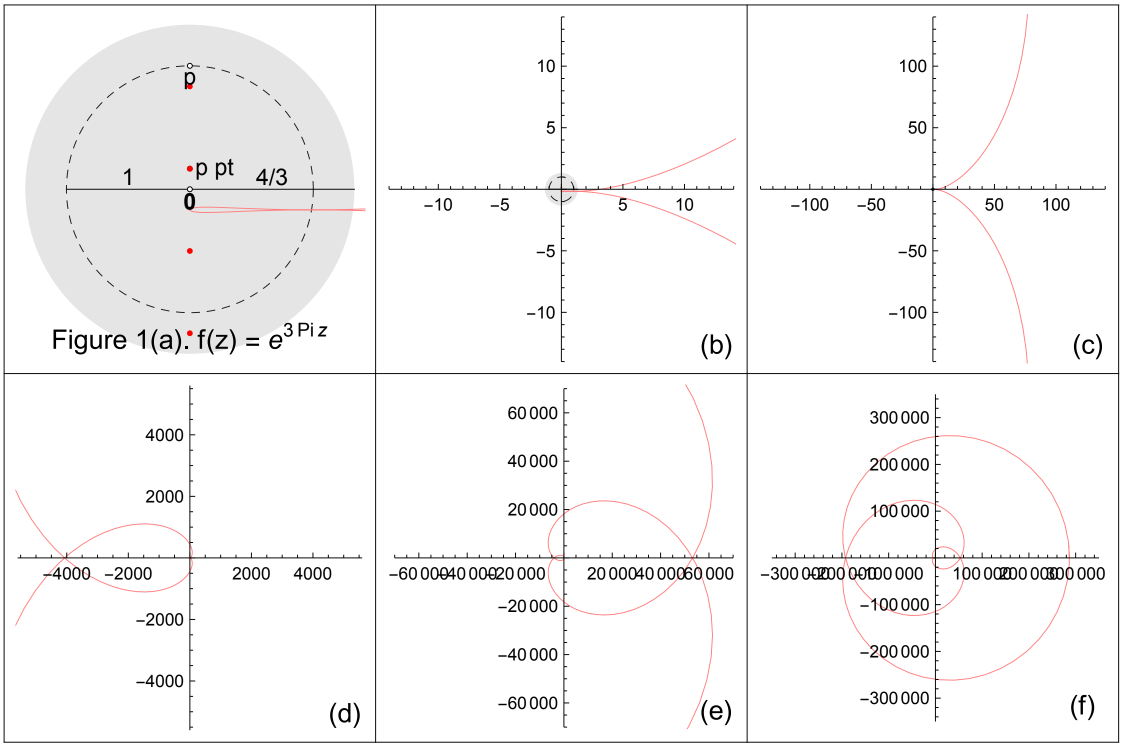


Figure 1(a): f(z) = $e^{3 \text{Pi} z}$

3 For each of the following functions f(z), find all the p-points lying inside the specified disc, determine their multiplicities, and by using a computer to draw the image of the boundary circle, verify the Argument Principle.

(i) f(z) = $e^{3\pi z}$ and p = i, for the disc |z| ≤ (4/3).

See Figure 1a, which shows f(z) on |z| ≤ 4/3, and figures (b)-(f), which zoom out successively from a.

f(z) = $e^{3\pi z}$ is analytic because exponential functions are analytic. For our purposes Γ is a circle defined by |z| ≤ 4/3. Calculate the p-point(s).

$$f(a) - p = 0$$

$$f(a) - i = e^{3\pi a} - i = 0$$

$$3\pi a = \text{Log}(i) = (\pi/2)\, i$$

$$a = (\pi/2)\, i / 3\pi = i/6$$

$$e^{3\pi a} = i$$

Multiplying f(a) by $e^{2\pi i}$, we get

$e^{3\pi a}e^{2\pi i} = i$

$3\pi a + 2\pi\, i = (\pi/2)\, i$

$3\pi a = (\pi/2)\, i\ - 2\pi\, i$

$a = i\,\pi\,(1/2 - 2)/3\pi = i\,(-3/2)/3 = -\, i/2$

$e^{3\pi a}e^{4\pi i} = i$

$3\pi a + 4\pi\, i = (\pi/2)\, i$

$3\pi a = (\pi/2)\, i\ - 4\pi\, i$

$a = \pi i\,(1/2 - 8/2)/3\pi = -(7/6)i$

We can write the equation $z = \frac{(\pi/2)\, i \pm 2\, n\, \pi\, i}{3\,\pi}$, where n = 1, 2, ...∞, which produces four p-points with lengths ≤ (4/3), indicated by red dots in Figure 1(a).

$\frac{5\, i}{6},\ \frac{i}{6},\ -\frac{i}{2},\ -\frac{7\, i}{6}$

These have lengths

0.833333, 0.166667, 0.5, 1.16667

Since f(z) is analytic, v[f, 0] = +1 for each p-point. The four p-points internal to the disc give f(z) a topological multiplicity of 4.

The Jacobian also shows that f(z) is analytic, because $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. Use the sign of the Jacobian to verify the topological multiplicity.

$f(z) = e^{3\pi z} = e^{3\pi(x+yi)} = e^{3\pi x}\, e^{3\pi y i}$

$= e^{3\pi x}\,(\cos(3\pi y) + i\,\sin(3\pi y))$

$= e^{3\pi x}\,\cos(3\pi y) + e^{3\pi x}\,\sin(3\pi y)\, i$

Let $u = e^{3\pi x}\cos(3\,\pi y)$ and $v = e^{3\pi x}\sin(3\pi y)$.

$\partial_x u = 3\pi\, e^{3\pi x}\cos(3\pi y) \qquad \partial_y u = -3\pi\, e^{3\pi x}\sin(3\pi y)$

$\partial_x v = 3\pi\, e^{3\pi x}\sin(3\pi y) \qquad \partial_y v = 3\pi\, e^{3\pi x}\cos(3\pi y)$

From (9), the topological multiplicity is the sign of the determinant of the Jacobian, which is calculated as on p. 310.

$$\det(f(z)) = (3\pi\, e^{3\pi x} \cos(3\pi y))\,(3\pi\, e^{3\pi x}\cos(3\pi y)) - (3\pi\, e^{3\pi x}\sin(3\pi y))\,(-3\pi\, e^{3\pi x}\sin(3\pi y))$$

$$= 9\,\pi^2\, e^{6\pi x} \cos^2(3\pi y) + 9\,\pi^2\, e^{6\pi x}\sin^2(3\pi y)$$

$$= 9\,\pi^2\, e^{6\pi x}$$

Then

$$\nu(a) = +1$$

This appears to agree with the Argument Principle, as N = v(a) = 1.

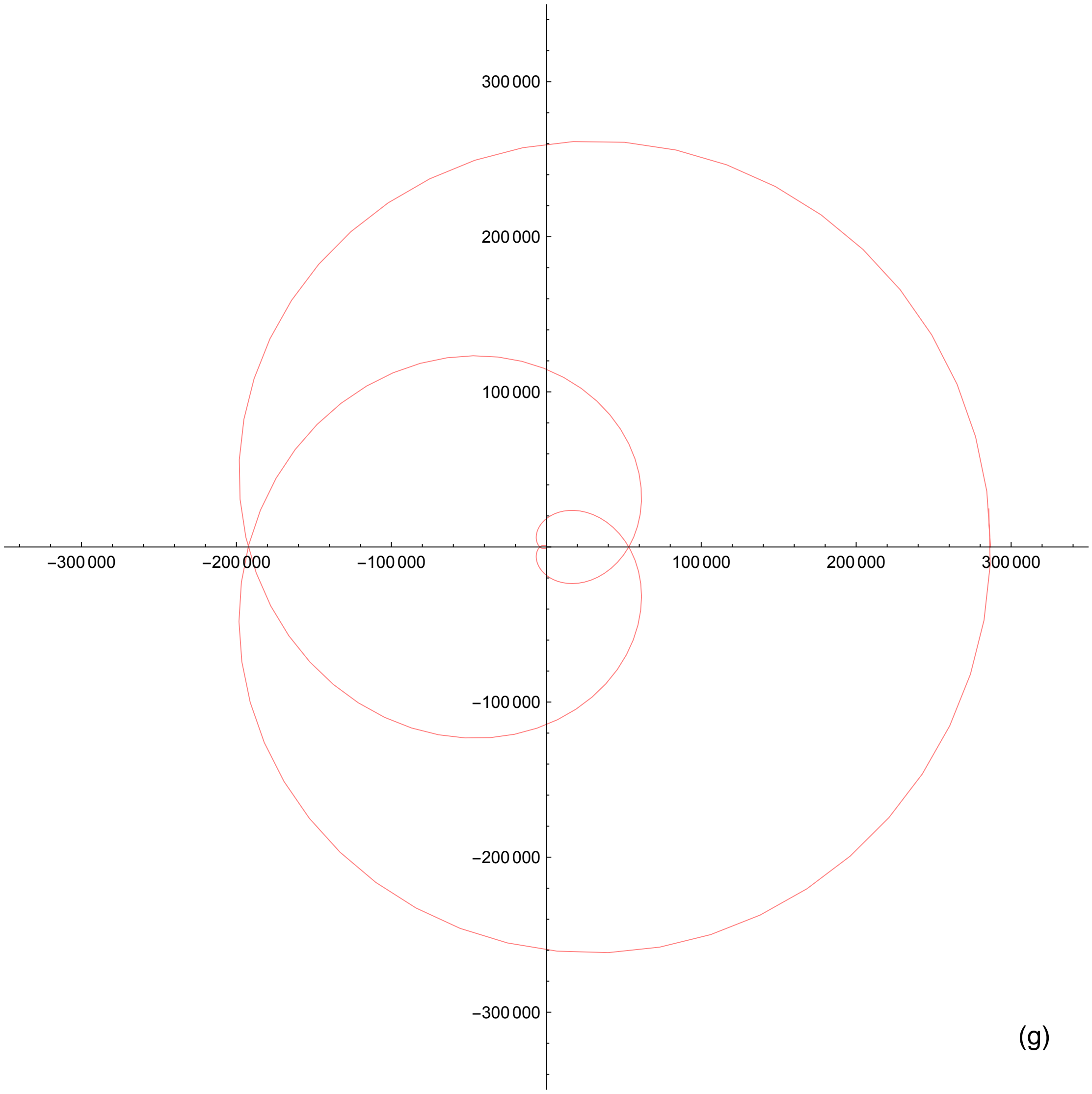
We can calculate the algebraic multiplicity for the first p-point as on p. 347 (top), as the power of Δ in the first non-zero derivative of the Taylor series at a.

$$f(a)\text{-}p = \frac{f^{(1)}}{1!}\Delta \ldots = 3\,\pi\, e^{3\,\pi\, a}\Delta \ldots$$

$$\simeq 3\,\pi\, e^{3\,\pi\,(i/6)}\Delta = 3\,\pi\, e^{i\pi/2}\Delta = 3\,\pi\, i\,\Delta$$

The first derivative at a is never 0 for any of the p-points, so the power of Δ is 1, giving an algebraic multiplicity of +1. This also agrees with the analytic topological multiplicity and the Jacobian topological multiplicity.

The low resolution of the figures makes it difficult to determine winding number by visual inspection, but it appears that a ray from the p-point in Γ to a point q outside the loop has four points of intersection and the loop appears to make four winds. If we combine the fact of N = 4 p-points in Γ with our calculation of algebraic multiplicity of +1 for each p-point and the visual observation of 4 winds in f(D), then N = *v*[f(D), p], which agrees with the Argument Principle. The curious lobe (Figure 1a) beneath the origin and the p-point is not counted in the winding and does not seem to affect the direction of the winding about 0.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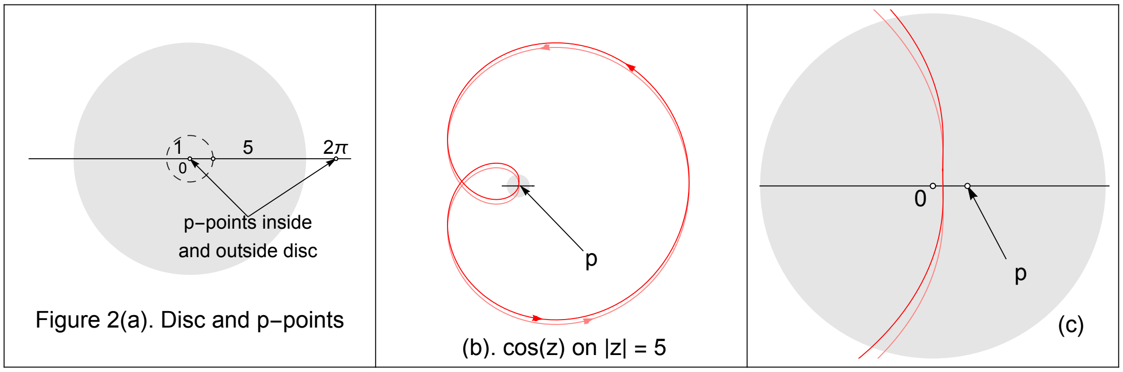

Figure 2(a). Disc and p–points (b). cos(z) on |z| = 5 (c)

(ii) f(z) = cos z and p = 1, for the disc |z| ≤ 5.

$$f(a) - p = \cos(a) - 1 = 0$$

$$a = 0, \pm 2\pi, \pm 4\pi, \ldots$$

The only p-point inside the disc is a = 0. Trig functions are analytic, so cos(z) is analytic. f'(0) = -sin(0) = 0, so a is a critical point and *v*(a) cannot be +1 (p. 350). From our understanding of the problem and of cos(z), we can assume that z and f(z) are both rotating in the positive direction and that z makes one trip around the disc. Since a is a critical point, we use the Jacobian or the Taylor's series to obtain the algebraic multiplicity. The Jacobian can not be applied where f'(a) = 0, so we apply the Taylor's series to find the first nonzero derivative of f(a).

$$f'(a) = -\sin(a)(\Delta - a) = -\sin(0)\Delta = 0$$

$$f''(a) = -\frac{\cos(a)\,(\Delta - a)^2}{2!} = -\frac{\Delta^2}{2}$$

The power of Δ in the first nonzero derivative is 2, so the algebraic multiplicity is 2.

In a first plot, the winding number as determined by inspection of (b) and (c) appeared to be 1, which would be a contradiction with the algebraic multiplicity. Vasco has provided the solution. He found that cos(z) makes two congruent winds around p for every trip of z around the disc. In 2(b) and (c) the second wind of cos(z) is offset by just enough to separate and reveal the two winds. Then N = *v*[f(D), p] = 2, which agrees with the Argument Principle.

To understand why one wind of z produces two winds of f(z), consider the location of f(z) after a half wind of z. We know that the cosine proceeds in a positive direction without reversing under a rotation of z. If f(z) makes a full wind and returns to its origin after a positive half wind of z, then it will make two winds after a full wind of z.

$f(z) = \cos(z) = \cos(\,|\,z\,|\,e^{i\alpha})$, where α can be thought of as the initial rotation angle.

$$\cos(\,|\,z\,|\,e^{i(\alpha+\pi)}) = \cos(\,|\,z\,|\,e^{i\alpha}\,e^{i\pi}) = \cos(-\,|\,z\,|\,e^{i\alpha}) = \cos(\,|\,z\,|\,e^{i(\alpha+\pi)})$$

So cos(z) is periodic with a period of π. That is, cos(z) makes a full wind for every half wind of z. When we send z around the disc for a full wind, cos(z) undergoes two full winds.

Figure (c) zooms in on (b).

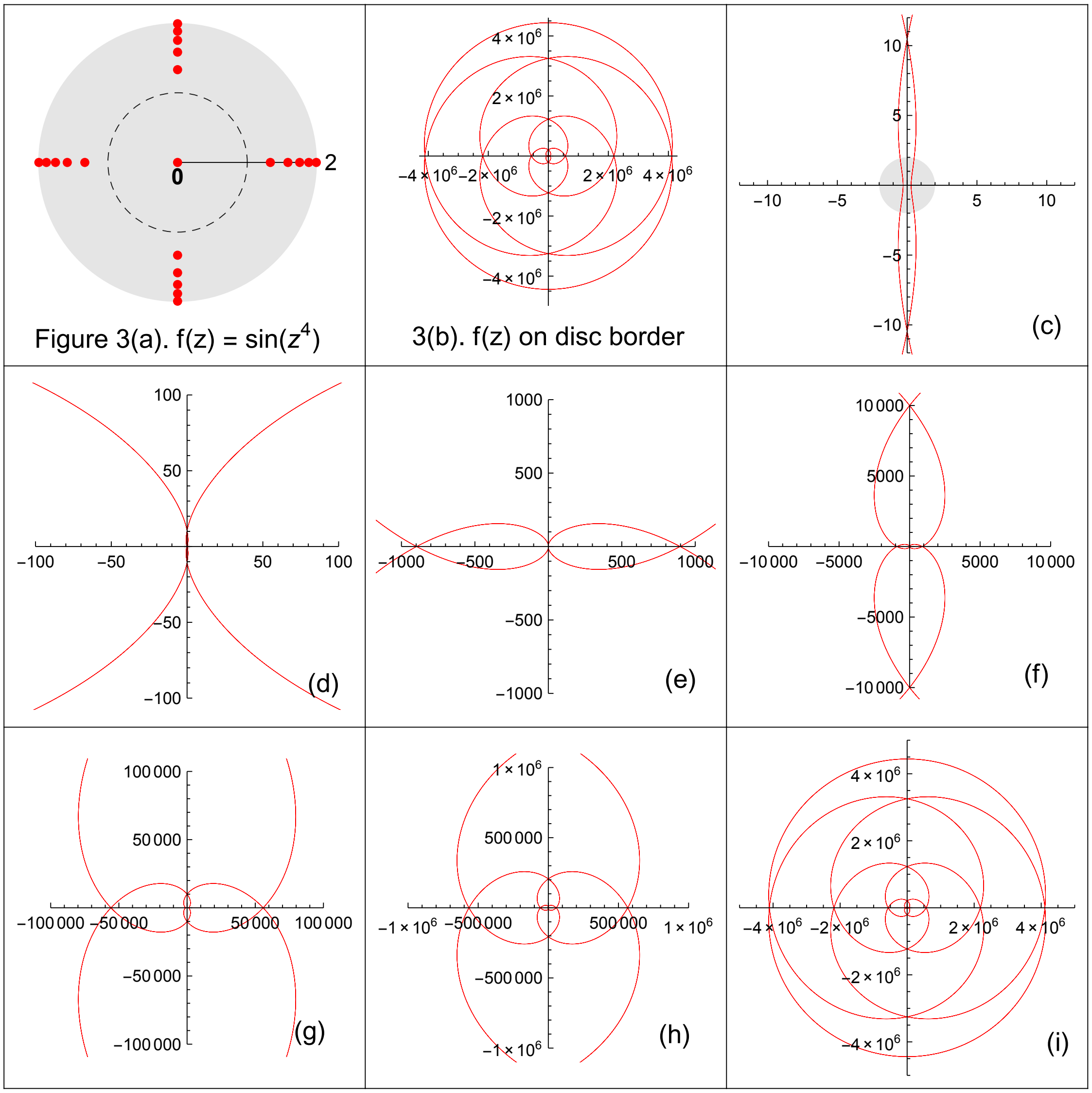

Figure 3(a). f(z) = sin(z^4)

3(b). f(z) on disc border

(iii) f(z) = sin z^4 and p = 0, for the disc |z| ≤ 2.

Calculate the topological multiplicity *v*(a) and determine the winding number *v*[*f*(Γ), p].

Let Γ be the circumference of the disc.

f(z) = sin z^4 is analytic because compositions of analytic functions are analytic. z^4 is analytic because it is a polynomial and sin() is analytic because it is a trig function.

$$f(a) - p = \sin a^4 - 0 = 0 \quad \Longrightarrow \quad a^4 = m\,\pi, \text{ where m is an integer}$$

$$a = \sqrt[4]{m\,\pi}$$

All of the roots must have the absolute value of any 4th root of mπ. The largest root of mπ is $(5\,\pi)^{1/4}$.

Then there are nonzero roots for m = -5..-1 and 1..5. Each of the $m\pi$ have four fourth roots. The four roots at each m must be $\{+(m\pi)^{1/4}, -(m\pi)^{1/4}, +i\,(m\pi)^{1/4}, -i\,(m\pi)^{1/4}\}$. Therefore, there are 40 nonzero fourth roots plus the four fourth roots of 0. In all there are 44 fourth roots, and each root is a p-point. Therefore, the total $v(a) = 44$. We get a winding number of $v[f(\Gamma), p] = 44$, because for analytic functions algebraic multiplicity n is $v[h(\Gamma_a)],\ p]$; v(a) = n, i.e. v(a) = algebraic multiplicity (p. 348). At this point, we have proven that $v_{total}(a) = v[f(\Gamma), p] = 44$. We could say “Done!”, but we check the work against calculations of algebraic and topological multiplicity as well as against the plot.

Calculate the algebraic multiplicity N.

Use the Taylor series to find the first non-zero derivatives for each p-point.

$$f(z) - p = f(a - \Delta) = 4\,a^3 \cos(a^4)\Delta - 6\,a^6 \sin(a^4)\,(\Delta - a)^2 - 8\,a^9 \cos(a^4)(\Delta - a)^3 + \ldots$$

m	$f^{(n)}(a)$
0	a = 0 is a critical point.
1..5	$f^{(1)}(a) = 4\,((m\pi)^{3/4})\cos(m\pi)\,\Delta = \pm 4\,(m\pi)^{3/4}\,\Delta \neq 0$
-5..1	$f^{(1)}(a) = 4\,(i^3(m\pi)^{3/4})\cos(m\pi)\,\Delta = \mp 4\,i(m\pi)^{3/4}\,\Delta \neq 0$

The 40 p-points not equal to zero each have a multiplicity of + 1 because $f^{(1)}(a) \neq 0$ at each of these points. The algebraic multiplicity of f(z) for each of the the 40 nonzero roots equals the power of $\Delta = (z-a)$, which is n = + 1 for each root, 40 for all the roots plus 4 for the critical point. We again conclude that N = 44.

Calculate the algebraic multiplicity of the zero point

The zero point is a critical point, so the Jacobian can not be used to determine multiplicity. The polynomial form of the Taylor’s series is $\sin(z^4) = z^4 - \frac{z^{12}}{3!} + \frac{z^{20}}{5!} + \ldots$ The first four derivatives for a = 0 are

$$f' = 4\,z^3\Delta - \frac{12\,z^{11}}{3!}\Delta + \ldots$$
$$f^{(2)} = (12\,z^2\,\Delta^2 - \frac{132\,z^{10}}{3!}\Delta^2 + \ldots)/2!$$
$$f^{(3)} = (24\,z\Delta^3 - \frac{1320\,z^9}{3!}\Delta^3 + \ldots)/3!$$
$$f^{(4)} = (24\,\Delta^4 - \ldots)/4! \simeq \Delta^4$$

We check with $f^{(4)}$ of the trig function:

$$f' = 4\,z^3\cos(z^4)\,\Delta$$
$$f^{(2)} = \frac{4}{2!}(\,3\,z^2\cos(z^4) - 4\,z^6\sin(z^4)\,)\,\Delta^3$$
$$f^{(3)} = \frac{4}{3!}\,(6z\cos(z^4) - 3\,z^2 4\,z^3\sin(z^4) - [24z^5\sin(z^4) + 16\,z^9\cos(z^4)])\Delta^4$$

Only the first term in parentheses becomes nonzero in f’’’’

$$f^{(4)} = (24\cos(z^4)\,\Delta^4 + 24z\,(-\sin(z^4)\,\Delta^4 = 24\Delta^4 + \ldots)/4!$$

$\simeq 24\ \Delta^4$

Both calculations go to Δ^4 + ..., which indicates an algebraic multiplicity of +4 for the critical point. This is in agreement with the idea that 0 has four fourth roots. Add these to the algebraic multiplicity of +40 for the 40 nonzero p-points to make a total algebraic multiplicity of N = 44.

These calculations of algebraic multiplicity provide further confirmation of the Argument Principle works, that N = ν[f(Γ), p] for this analytic function.

Determine the winding number $\nu[f(\Gamma), p]$ by tracing the plot

Working with the computer generated plot itself, one finds that the full 11 winds can be produced as z traverses an angle of only $\pi/2$. This suggests that the periodicity of $\sin(z^4)$ is actually $\pi/2$ as Vasco proposed, which is twice the periodicity of the cos(z) (See ii). It means that the observed number of 11 windings is repeated four times as z goes once round Γ making a total of 44, which equals the winding number determined in the first paragraph of this answer.

The 11 winds were counted by blowing up images (c) to (i) to full page size and tracing loops while numbering end points and crossings of the real line to the left of my starting point, which was the point closest the origin on the negative real axis. It would have been more orthodox to begin on the positive real axis. The plot is symmetrical, so one could remove the minus signs from the positions in the table and get the result for the orthodox starting position. The starting point is visible in (c). The eleven crossing points are arranged like a palindrome. They proceed outwards from the starting point at ≃ .3 to -4.2 x 10^8, then track back to -1000, then expand again to -4.2 x 10^8, and finally shrink and return to the start.

Approximate position of numbered crossings given starting point closest the origin on the negative real axis.

N	**position**
0, 11	-.3 (starting and ending point)
1, 10:	-51,000
2, 9	-2.4×10^6
3, 8	-4.2×10^8
4, 7	-575,000
5, 6	-1000

The essential calculations in the computer program are as follows:

zPts = Table(2 Exp[I θ], {θ, 0,2π, .001}) # 1000 points in a circle of radius 2

f(z) = $\sin(z^4)$ applied to all z in zPts

Figure (c) zooms in on the center of (b), Figures (c-h) zoom out by one order of magnitude each, and then (i) zooms out by 4 times. (j) blows up (i). The small gray box in (j) corresponds to (g).

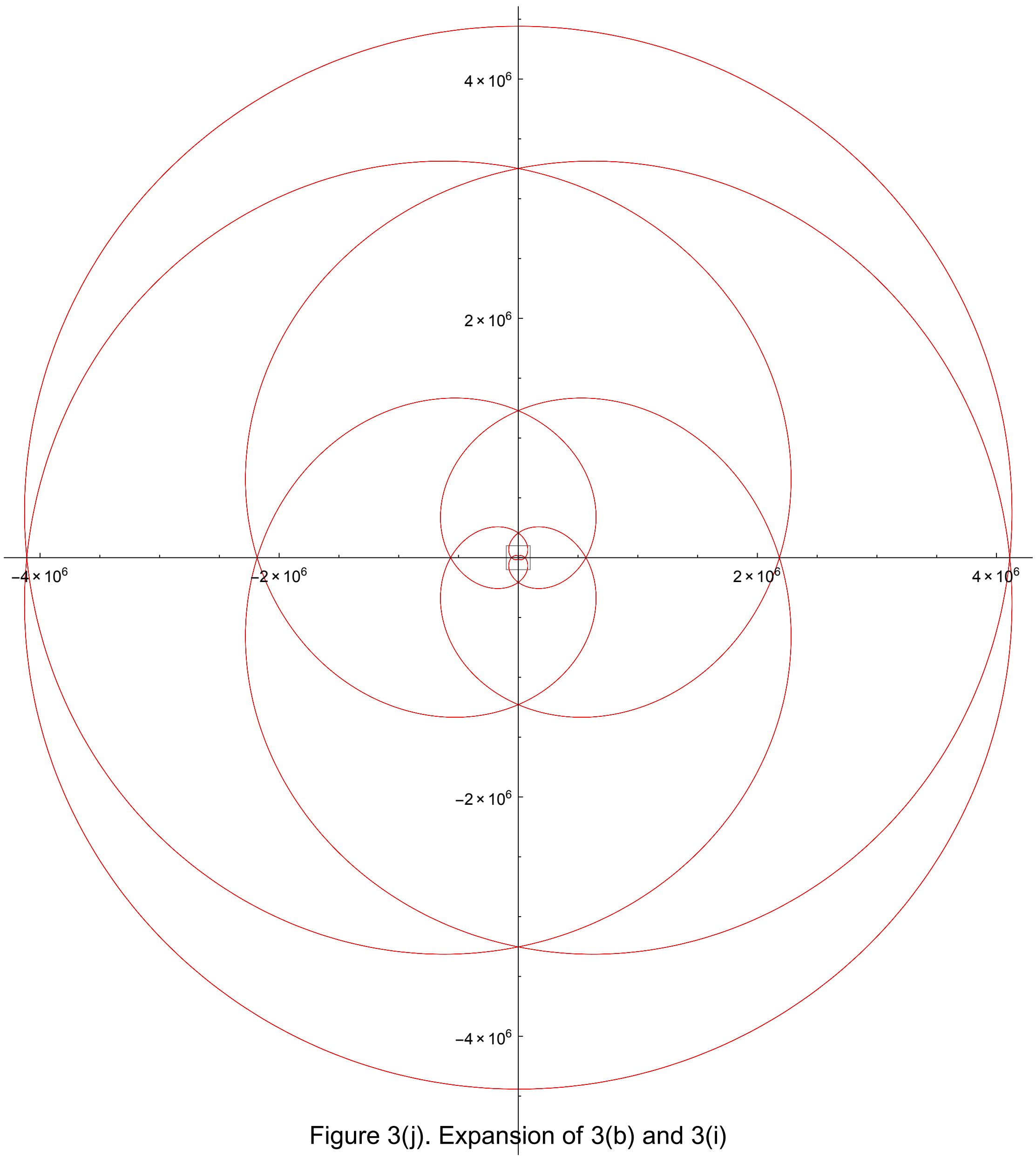

Figure 3(j). Expansion of 3(b) and 3(i)