Needham, Chapter 7, Exercise 22, p. 375-376.
G. Palmer, October 4, 11:30 am PST

22 In [11] we used the idea of deformation to derive the argument principle. The figure below shows another method. The interior of Γ, containing various p-points and poles, has been crudely partitioned into cells C_j in such a way that each one contains no more than a single p-point or a single pole. Now think of each cell as being a loop traversed in the conventional direction; this sense is indicated for two adjacent cells in the figure.

[Hint: If an edge of a cell does not form part of Γ then it is also an edge of an adjacent cell, but traversed in the opposite direction. What is the net rotation of f(z) round p as z traverses this edge once in one direction, then in the opposite direction?]

(i) What is the value of $\nu\left[f\left(C_j\right),\ p\right]$ if C_j (1) is empty; (2) contains a p-point of order m; (3) contains a pole of order n?

(1) 0
(2) m
(3) -n

This is by disregarding the counter rotations of the empty cells, or as Vasco put it (Forum), "If we draw each C_j separately on the plane, the image loop appears in the image plane and we can see how it winds round p. We can do this for each C_j separately."

(ii) Obtain the Argument Principle by showing that

$$\Sigma_j\, \nu\left[f\left(C_j\right),\ p\right] = \nu[f(\Gamma), p]$$

Based on my answer to (i), I am adopting the position that Needham intended for the C_j on the LHS to be considered independently of their neighbors. It is only in finding ν[f(Γ), p] that we apply the principle that traversals of boundaries between neighboring cells are cancelled.

Find the value of $\boldsymbol{\Sigma_j\, \nu\left[f\left(C_j\right),\ p\right]}$. If we take each cell C_j and disregard the countervailing rotations on internal boundaries, then for i, k, and l in the set of j and i ≠ k ≠ l, where m and n are winding numbers (i.e. multiplicities) for f on a complete traversal of z round a cell or loop containing a p-point (m) or a pole (n),

$\nu[f(C_i),\ p] = m_i$ for a cell containing a p-point
$\nu[f(C_k),\ p] = -n_k$ for a cell containing a pole
$\nu[f(C_l),\ p] = 0$ for cells not containing either a p-point or a pole

Then the sum of the winding numbers is

$$\Sigma_i\, m_i - \Sigma_k\, n_k + \Sigma_l\, 0 = \text{M} - \text{N}$$

where M and N are the number of winds of all the p-points and poles in all the cells, counted with their multiplicities (def. p. 364). Then $\Sigma_j\, \nu\left[f\left(C_j\right),\ p\right]$ = M - N.

Find the value of ν[f(Γ), p]. The interior cells in the figure have no p-points or poles. The only segments which have no cancelling traversals in the opposite direction are the boundary segments on Γ. Now to obtain a complete wind about every point, z traverses just the segments along the whole boundary resulting in f(Γ). Every boundary segment contributes some degree of rotation around all of the p-points and poles. Adding up fractional rotations contributed by boundary segments, the total for the closed loop Γ is 2π. The rotation of **f(z)** on Γ is applied via traversal of z on Γ with consideration of all the p-points and poles inside Γ counted with their multiplicities m and -n. The winds of f(Γ) add up to M-N for one full wind of z around Γ.

We index the boundary segments s_i, p-points with m_j, and poles with n_k. Let R(Γ) be the net rotation of f(z) as z traverses Γ once. R(Γ) = $\Sigma_i\, R(s_i)$. We introduce the notation $R(\Gamma,\ m)$ to represent the rotation of f(z) round a point p inside f(Γ) when z traverses Γ round a p-point of multiplicity m, with a parallel notation for poles. $R(\Gamma,\ m) = 2\,\pi\, m$. The net rotation of f(Γ) around p when z traverses Γ is $\Sigma_j\, R\left(\Gamma,\ m_j\right) - \Sigma_k\, R(\Gamma,\ n_k)$. This quantity is divided by 2π to obtain the sum of the winding numbers.

$$\nu[\mathrm{f}(\Gamma), \mathrm{p}] = \frac{1}{2\pi}\ [\Sigma_j \mathrm{R}(\Gamma, m_j) - \Sigma_k \mathrm{R}(\Gamma, n_k)]$$

$$= \frac{1}{2\pi}\left[\Sigma_j\, 2\,\pi\, m_j - \Sigma_k\, 2\,\pi\, n_k)\right]$$

$$= \sum_j \mathtt{m_j} - \sum_k n_k = \mathrm{M - N}$$

thereby obtaining the argument principle and establishing that $\Sigma_j\, \nu\left[f\left(C_j\right),\ p\right] = \nu[\mathrm{f}(\Gamma), \mathrm{p}]$.

This works because m and n are properties of f(z) on a closed loop. In both situations, z makes one traversal about each p-point and pole while f(z) makes m or n winds about p for each p-point or pole.

Note: Needham reversed the assignments of m and n compared to M and N to p-points and poles. I have assigned m and M to p-points and n and N to poles.