Needham, Chapter 7, Exercise 21, p. 375
G. Palmer, October 26, 2017, 9:30 am PST

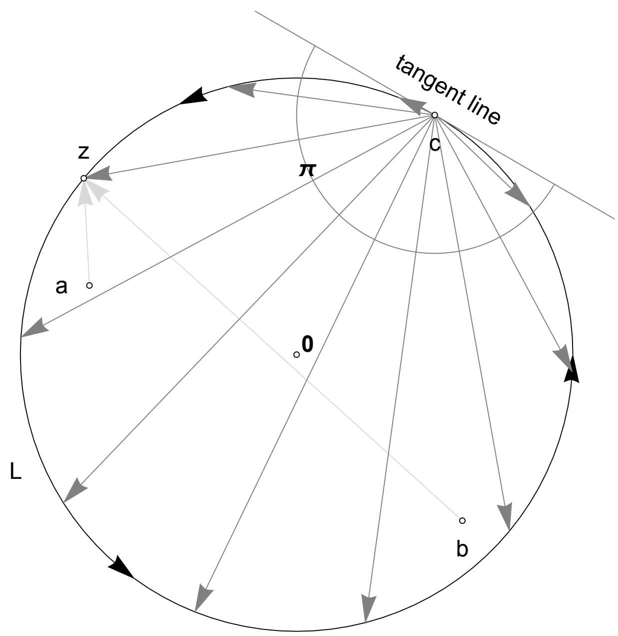

Figure 1: Net rotation of f(z) about root on L

21 Note: This exercise is rewritten according to Vasco's suggestions. Γ is substituted for L and sentence 2 is rewritten. $\nu[L, p]$ now reads *$\nu[f(\Gamma), p]$. See the errata for chapter 7 on the forum.*

21 Let $\mathcal{R}(\Gamma)$ be the net rotation of f(z) round p as z traverses a loop L. For example, if none of the p-points lie on Γ, then p does not lie on $f(\Gamma)$ either. Then

$$\nu[f(\Gamma), p] = \frac{1}{2\pi}\mathcal{R}(\Gamma)$$

By taking this formula to be the definition of ν, make sense of the statement that the Generalized Argument Principle (17) remains valid even if there are some poles and p-points on Γ, provided that we count these points with half their multiplicities.

Answer

Note that $\mathcal{R}(\Gamma)$ is defined as rotation of f(Γ) about a point p and that $\nu[f(\Gamma), p]$ is defined in terms of $\mathcal{R}(\Gamma)$.

"Net" revolution is defined and described in ¶ 1 of I Winding Number, p. 338. For roots located in the interior of a loop Γ, the net number of revolutions of $f(\Gamma)$ around p equals the multiplicity of the root (Exercise 4, and animation, p. 365*). In case Γ touches a p-point (of multiplicity 1), the net rotation of f(Γ) about p is only π, half a revolution, as shown in Figure 1 (and implicit in exercise 8), where the rotation of f(Γ) about c is swept out by the gray arrows from c to points on Γ. In traversing Γ, z sweeps out an angle from arg(ξ) at c to arg(-ξ), where ξ is a tangent vector at c. If the p-point is a pole rather than a root, the same principle applies, but the angle swept out is -π.

The net number of complete revolutions is the winding number. The discussion on p. 339 indicates that only complete revolutions are counted. There, a revolution of π or -π of f(L) as z traverses L counts as ν[f(L), p] = 0 (Section 2, ¶ 3) not 1.

But if the Generalized Argument Principle is to be valid for functions with roots and poles situated **on** Γ, we divide their multiplicities by two. For these roots and poles, each traversal of z around Γ represents half a revolution, either π for a root or -π for a pole. Then for roots and poles combined **on** Γ is ν[f(Γ), p] = N - M, where N is the number of roots inside and on Γ and M is the number of poles inside and on Γ counted with their multiplicities and divided by two where they lie on Γ.

Example.

A simple example is an analytic function f(z) = z (z+1) with two roots 0 and -1, of which -1 lies on the unit circle C (Figure 2), and so its multiplicity must be divided by 2. The two roots 0 and -1 of f both have multiplicity 1. Both f(0) and f(-1) lie on f(C) but we are not concerned with that because the point of the GAP is to calculate the rotations of f(Γ) from the qualities of its p-points and poles. Furthermore, 0 is not a p-point *on* Γ of f(0), because 0 does not lie on the loop Γ. Since we have only p-points and no poles, we have only to find N = $\frac{1}{2\pi}[\mathcal{R}(\Gamma)_0 + \mathcal{R}(\Gamma)_{-1}]$. Then, by this generalized version of the GAP, ν[f(Γ), p] is defined as $\frac{1}{2\pi}\mathcal{R}(\Gamma)$, so

$$\nu[f(\Gamma), p] = \nu[f(\Gamma), 0] = \frac{1}{2\pi}\mathcal{R}(\Gamma) = \frac{1}{2\pi}[\mathcal{R}(\Gamma)_0 + \mathcal{R}(\Gamma)_{-1}] = \mathrm{N}$$

$$= \frac{1}{2\pi}\left[\, 2\pi\,\nu(0) + 2\pi\,\frac{\nu(-1)}{2}\,\right] = \frac{1}{2\pi}\left[\,(2\pi \cdot 1) + (2\pi \cdot \tfrac{1}{2})\,\right] = \frac{3}{2}$$

But what does the fractional value mean? Are we calculating fractional winds, or just fractional rotations? Vasco insists that the problem requires reformulation of the GAP as $\mathcal{R}(\Gamma) = 2\pi(N-M)$, which is only implicit in the above. The Wikipedia page uses only whole numbers for winds. Physicists may use fractional winds. But why should there not be fractional winds?

In $\nu[f(\Gamma), p]$, p = 0 because both p-points are roots, meaning that f(p) = 0 for both 0 and -1.

* <faculty.unlv.edu/gbp/mma/VisualComplexAnalysis/CHAPTER7/>.

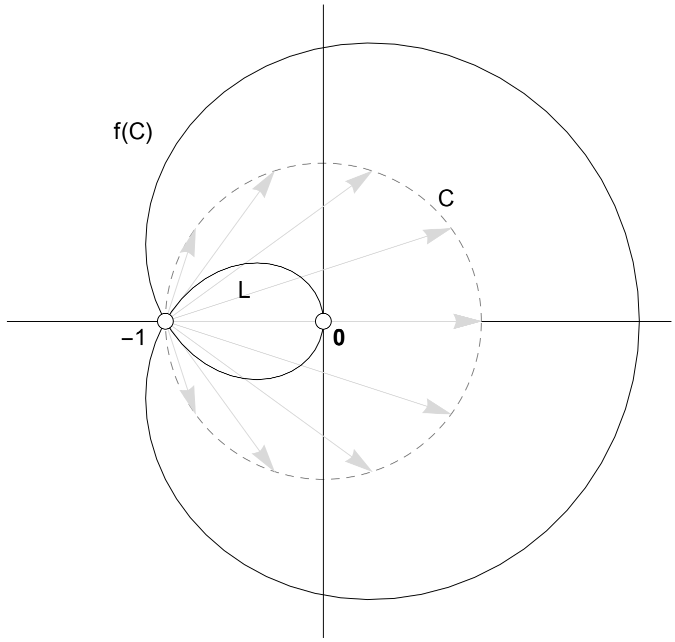

Figure 2. Two p−points (−1) and 0
with images on C and f(C)