Needham, Chapter 7, Exercise 20, p. 374-5
G. Palmer, June 22, 2017, 6:20 am PST

20 Let f(z) and g(z) be analytic inside and on a simple loop Γ. By applying the Maximum Modulus Theorem to (f-g), show that if f = g on Γ then f = g throughout the interior.

The Maximum Modulus Theorem (p. 355) says that the maximum of $|f(z)|$ on a region where f is analytic is always achieved by a point on the boundary of Γ; never by a point inside.

If $f = g$ on Γ, then $f-g$ is analytic and $f-g = 0$ on Γ, so the maximum of $|f-g|$ on the region enclosed by Γ is 0. Hence, $f-g = 0$ for all p in the region enclosed by Γ, and $f = g$ throughout the interior.

An alternative proof that does not use the MMT.

If $f = g$ on Γ, then $f' = g'$ on Γ, and at any p on Γ, the amplitwist $f'(p) = g'(p)$. Then for any infinitesimal movement ξ at p, $f(p)+f'\xi = g(p)+g'\xi$. If one always gives ξ the direction -p, then by applying f(p) to successive points on the ray p, as $|p| \longrightarrow 0$, $f(p)+f'\xi = g(p)+g'\xi$ at all points on the ray. Since this is true for all p in Γ, then $f = g$ throughout the interior.