Needham, Chapter 7, Exercise 2, p. 369
G. Palmer, March 24, 2017, 4:15 pm PST

2 Reconsider the mapping $\hat{\mathcal{L}}$ in (4) of the unit circle to itself, and the associated graph of $\Phi(\theta)$ in [7]. If $\Phi'(a) > 0$ then the graph is rising above the point $\theta = a$, and [a] small movement of z will produce a small movement of the w having the same sense. We say that Φ is orientation-preserving at a and that the topological multiplicity $\nu(a)$ of $z = e^{ia}$ as a preimage of $w = e^{i\Phi(a)}$ is +1. Similarly, if $\Phi'(a) < 0$ then the mapping is orientation-reversing and $\nu(a) = -1$. In other words,

$\nu(a)$ = the sign of $\Phi'(a)$.

Compare this with the 2-dimensional formula (9).

In (9) the winding number v(a) of a under a continuous function equals the sign of the determiner of the Jacobian.

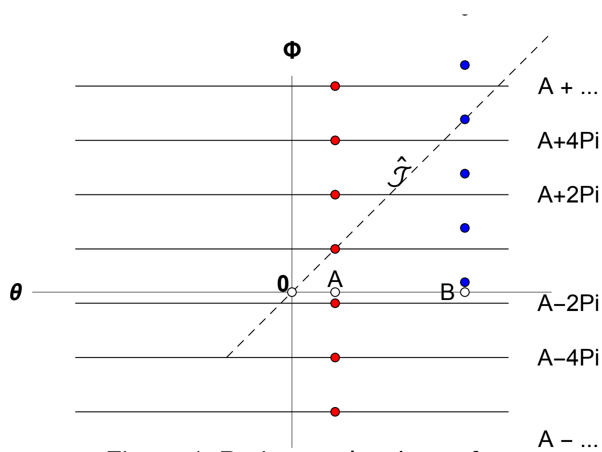

Figure 1. Preimage Φ values of $w = e^{iA}$ (red) and e^{iB} (blue)

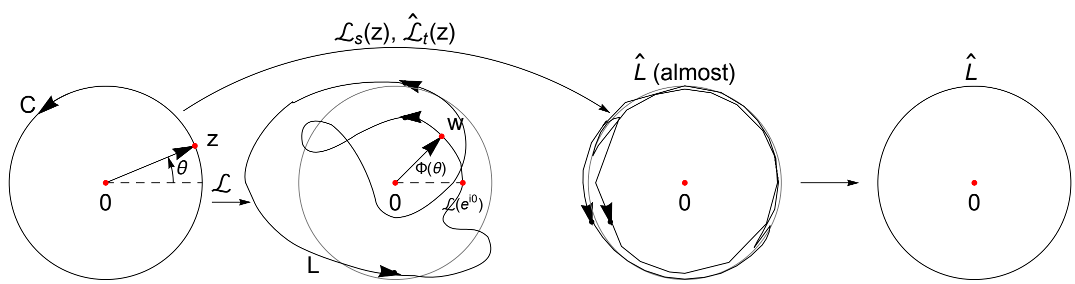

Figure 2. Loops as mappings of the unit circle

(i) In [7], explain how the complete set of preimages of $\hat{w} = e^{iA}$ can be found by drawing the family of horizontal lines $\Phi = A$, $A \pm 2\pi$, $A \pm 4\pi$, etc.

If we examine the generation of L in Figure 2, we can think of θ rotating counterclockwise on C and affecting Φ on L, while R(θ) moves the loop in and out with respect to 0 on L. The values of Φ in w = $\mathcal{L}$(z) = $R(\theta)\, e^{i\Phi(\theta)}$ are given as *Φ = A ± n2π, A ± 4π, etc*. If 0 <A ≤ 2π, n is the number of winds of L. Each of the Φ values represents a point on a loop superimposed on the unit circle. Substitute these values for Φ to get an infinite series of w's on L all aligned on a ray at angle A. In this answer, I am assuming that there are no variations in R(θ), no loops winding back and intersecting. Consequently, all such w's, no matter how many windings, fall on the same point of L, which is e^{iA}. All of these w's can be drawn at the same angle A in L. One could say "done", but it appears that the image belongs on $\hat{L}$, because Needham wrote "Reconsider the mapping $\hat{\mathcal{L}}$ in (4) of the unit circle to itself." I think all that needs to be done at this point is to rename L to be $\hat{L}$. There is no need to apply $\mathcal{L}_s$ or $\hat{\mathcal{L}}_t$, but we recognize that there must be a function $\hat{\mathcal{L}}$: L ⟶ $\hat{L}$. In this case, it is the identity function. L contains the preimages, $\hat{L}$ the image.

Each of the horizontal lines represents the angle A plus or minus n winds. The horizontal lines represent all the possible preimages for $\hat{w}$ where arg(z) = A.

(ii) If the set of preimages is typical in the sense that Φ' ≠ 0 at any of them, what do we obtain if we sum their topological multiplicities? Thus to say that the degree of $\hat{\mathcal{L}}$ (the winding number of L) is ν is essentially to say that $\hat{\mathcal{L}}$ is ν-to-one. [Hint: In [4], consider a ray as describing the location of w.]

Preimages with Φ' ≠ 0 have positive or negative slopes of Φ according to the direction of winding. Then $\hat{w} = e^{i\Phi(\theta)}$ and ν(a) = ±1. The multiplicities can be found as in Figure [4] by summing the number of times a ray anchored at 0 intersects a circle on L (assuming the loops have been drawn in and tightened to the unit circle). Figure [4] gives an indication of the situation, but in the present case, p = 0. The result is ±1 per intersection with the sign depending on the orientation of each circle. Figure [4] describes changes in multiplicity as the point p moves from the inside to the outside of a winding loop. In the present case, the rotation of the ray describes the location of w, p remains fixed at the origin, and the summation of segments intersected by the ray indicates the total multiplicity $\nu[L, 0]$ (the preimages after drawing but before shrinking to the circle). After shrinking, if any, if the ray intersects a preimage loop segment oriented from left to right, $\nu[\hat{L}, 0]$ decreases by n (the winding number of the preimage circle). If it intersects a preimage loop segment oriented from right to left, $\nu[\hat{L}, 0]$ increases by n. Of course, since all these preimages have been drawn congruent to the unit circle, it would intersect all the preimages and segments of preimages simultaneously. On the ray, Φ(θ) remains constant; $\nu[\hat{L}, 0]$ varies for each preimage. Since the ray intersects $\hat{L}$ only once **for each preimage** (each $\hat{w}$), we can say that $\hat{\mathcal{L}}$ (the function from L to $\hat{L}$) is ν-to-one. The total is $\sum_n \nu\left[\hat{L}_n, 0\right]$, the sum of the winding numbers for all the preimage loops.