Needham, Chapter 7, Exercise 19, p. 374-5
G. Palmer, May 22, 2017, 8:30 pm PST

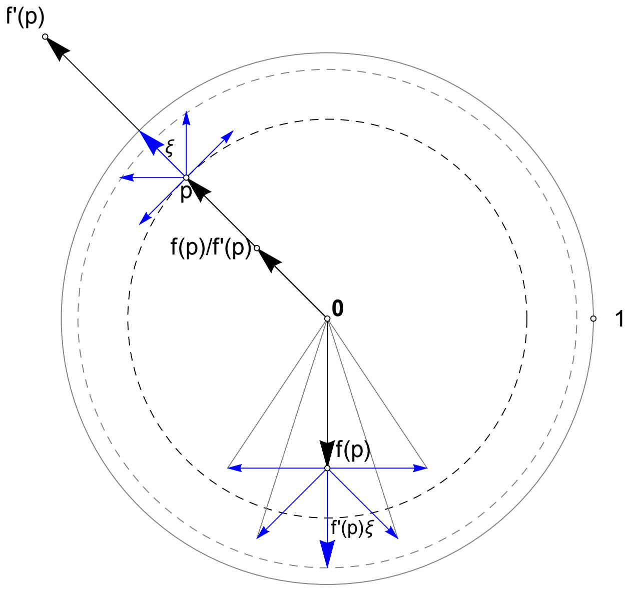

Figure 1: Direction [f(p)/f'(p)] for $\Delta|f|_{max}$, f=z^2

19 (i) If p is not a critical point of an analytic function f, show geometrically that the modulus of f increases most rapidly in the direction [f(p)/f'(p)].

The question doesn't specify whether it is f or z that is moving in the direction [f(p)/f'(p)]. In most cases, f and z would move in different directions. I will assume that it is z that is moving in the direction [f(p)/f'(p)] and that z has the initial value p.

The modulus increases with the distance of z from the origin. It increases most rapidly as z moves in the direction arg(p). In Figure 1, we can see that the rate of increase of the modulus in the direction of p is $|f(p+\xi)/f(p)|$. The rate of increase of the modulus in other directions must be less because movement of z in any direction but arg(p) produces a smaller value for |z|. The angles and relative magnitudes are preserved under an analytic f. The gray arrows in Figure 2 indicate absolute values of z under rotations of ξ, where $z = p + \xi\, e^{i \arg \phi}$, where ξ has direction arg(p), and ϕ is any angle at p. Figure 1 shows that all of the rotations away from arg(p) result in values of |f(p)| less than $|f(p) + f'(p)\,\xi|$. The ray $(f(p) + f'(p)\,\xi)$ touches the outer dashed circle, but the other rays do not. Consequently, the maximum increase in |f(z)| for infinitesimal changes $|\xi|$ in any direction equals $|f(p + \xi) - f(p)|$, which occurs in the direction p.

Now it remains to be shown that arg(f(p)/f'(p)) = arg(p).

arg(f(p)/f'(p)) = arg(f(p)) - arg(f'(p))

which is just the direction of f(p) minus the twist. The result is arg(p). In Figure 1, the angle of p and ξ is

$3\pi/4$ by construction. Under f = z^2, arg(f'(p)) equals arg(p), so the twist is also $3\pi/4$. The angle of f(p) = z^2 is (2 arg(p)) = $3\pi/2$. Then arg(f(p)) - twist = $3\pi/4$ = arg(f(p)/f'(p)).

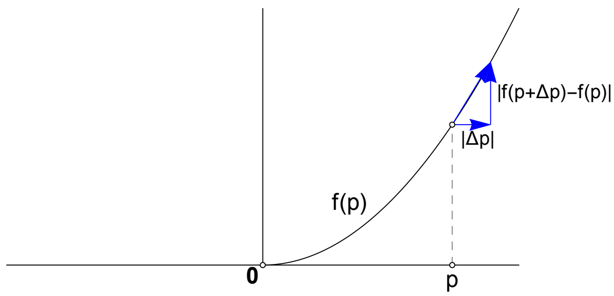

Figure 2: Slope of modular surface equal to |f'(p)|

(ii) In terms of the modular surface above p, this direction [f(p)/f'(p)] lies directly beneath the tangent line to the surface having the greatest "slope" (i.e., tan of the angle it makes with the complex plane). Show that the slope of this steepest tangent plane is |f'(p)|, and note the analogy with the slope of the ordinary graph of a real function.

This question strikes me as a bit vague and hard to parse because it shifts from "tangent line" to "tangent plane". Of course, the steepest tangent line to the modular surface lies in a plane tangent to the modular surface at the same point, and both line and plane have the same slope.

I think by "steepest slope", Needham does not mean the steepest slope anywhere in the modular surface. I think he means that of all the tangent lines one could plot from a point directly above p on the modular surface (and all lying in the tangent plane), the line that aligns with the direction of p is the steepest.

Figure 2 shows a plane vertical to the complex plane passing through the origin and through the point p, which lies somewhere in the complex plane. Figure 2 shows that the slope of f(p) is $\frac{|f(p+\Delta p)-f(p)|}{|\Delta p|}$ = |f'(p)|. By using absolute values, we can make the analogy to the slope y/x of the ordinary graph of a real function.

One could also write $\frac{|f'(z)|}{|z'|} = \frac{|f'(z)|}{1}$ = (|f'(z)| at p) = |f'(p)|.

(iii) What does the modular surface look like at a root of order n?

I take the order to be the multiplicity of the root. Vasco describes the function f = $(z - a)^n$g(z) as having a root of order n (Forum). The modular surface at a root appears as a cone, bowl, or funnel touching $\mathbb{C}$ and opening upwards. Roots with multiplicities greater than 1 expand or contract the modular surface more rapidly producing more gentle (expanded) or steeper (contracted) slopes.

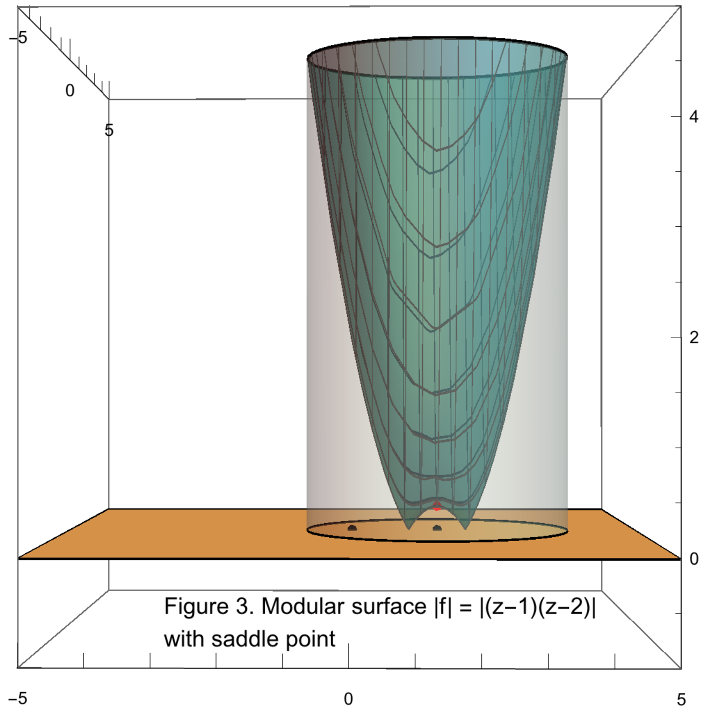

Figure 3. Modular surface |f| = |(z−1)(z−2)| with saddle point

(iv) What does the modular surface look like above a critical point of order m? Using the case m = 1, explain why such places are called saddle points.

A critical point is where f'(z) = 0. It is a point where conformality breaks down (p. 204). It is "locally crushing" (p. 205, 349) and, for analytical functions, perfectly symmetrical in all directions (p. 349). It is a zero in the case of power functions of the form z^n An angle between rays is multiplied by n at the critical point (p. 205). Except in the trivial case of f(z) = const., it is never a maximum, because there are no local maximums. It is a point where a cubic (analytic) function on a loop crimps, and it is the birth point of a new loop in f(Γ) as a Γ loop expands to include it [Exercise 4 (i), (ii), Ch. 7].

The problem statement appears to contradict the discussion of critical points on p. 205, § 2 Breakdown of Conformality, where we find "We quantify the degree of this strange behavior by saying that z = 0 is a critical point of order (m-1)." But (iv) above introduces a "critical point of order m" and "the case m = 1". To follow the logic and allow for instances in which the same symbol is applied to different objects according to circumstances, we conclude that order m = 1 could apply to a mapping $z^{m+1} = z^2$. We find confirmation on pp. 346-347. Regarding analytical mappings in general "if the order of the critical point is (n-1), so that $f^{(n)}$ is the first non-vanishing derivative at a..." then the algebraic multiplicity of a is defined to be n. So the order is one less than the algebraic multiplicity and one less than the order of the first non-vanishing derivative. This is a bit confusing, because a polynomial of order 3, for example, could have a critical point of order 2 and a non-vanishing derivative of order 3 which would be a polynomial of order 0, i.e. a constant. The orders of derivatives run in reverse direction to the order of polynomials, because derivatives have lower order than their integral functions. The higher the n in $P^{(n)}$, the lower the power of the derivative.

Needham's Figure [3], Chapter 2, § I, p. 57 shows that the modular surface above the critical point 0 of z^2 is a paraboloid. This is somewhat puzzling, because 0 doesn't look like a saddle point, but the problem says to explain why such points are called saddle points. Perhaps it is just a degenerate saddle point, because the only roots are 0 and the critical point is 0. In Figure 3 of this document, we see a small saddle centered at z = 3/2 for f = (z-1)(z-2), for which f′(z) = 0 has the solution z = 3/2. The saddle point of the modular surface has height |(1/2)(-1/2)| = 1/4.

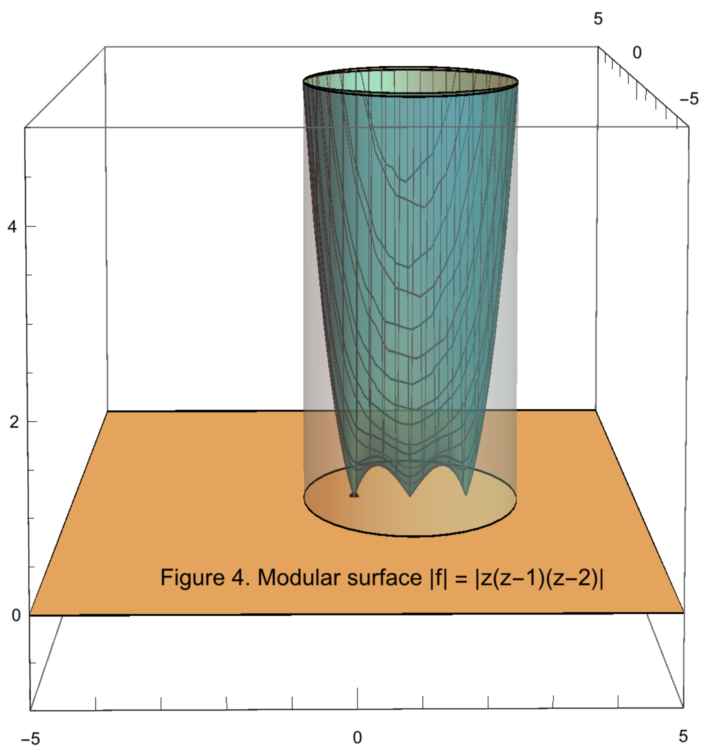

Figure 4. Modular surface |f| = |z(z−1)(z−2)|

(v) Rephrase MacDonald's Theorem [Ex. 13] in terms of the number P of pits and the number S of saddle points in the portion $\mathcal{A}$ of the modular surface lying above Γ and its interior.

MacDonald's Theorem: "f has one more root inside Γ than f′ has."

Figure 3 is the modular surface of f = $(z-1)(z-2)$, which has two roots. f has one critical point at z = 3/2. Accordingly, the revised MacDonald's theorem might read:

"An analytic f has one more pit than it has saddle points in the portion of $\mathcal{A}$ of the modular surface lying above Γ and its interior; P - S = 1."

What if f = z (z-1)(z-2), as in Figure 4? There are three pits (roots) and two saddle points (critical points). P-S = 1. Our topographical rephrasing holds for this simple example, but so does MacDonald's Theorem.

(vi) ...Now imagine a persistent rain falling on the [modular] *surface $\mathcal{A}$. The pits gradually fill with water and so become P lakes, the depths of which we shall imagine are always equal to each other. What happens to the number of lakes as the water successively rises past each of the S passes? How many lakes are left by the time that the water has finally risen to the level of K? As required, deduce that*

P = S + 1

All lakes separated by passes at the same level merge. The simplest case is when two lakes merge into one. At level K, all lakes have merged into one, which completely fills the modular basin. When the basin is filled, P_0 = 1 and S_0 = 0, so P = S + 1. If we were to drop the level to just below the highest saddle, there would be two lakes and one saddle. $P_1 = S_1 + 1$. If we were to drop the level below the next lower saddle, one would see three lakes and two saddles. Every time the water lowers below the level of a submerged saddle, one lake becomes two, so it is always true that P = S + 1.

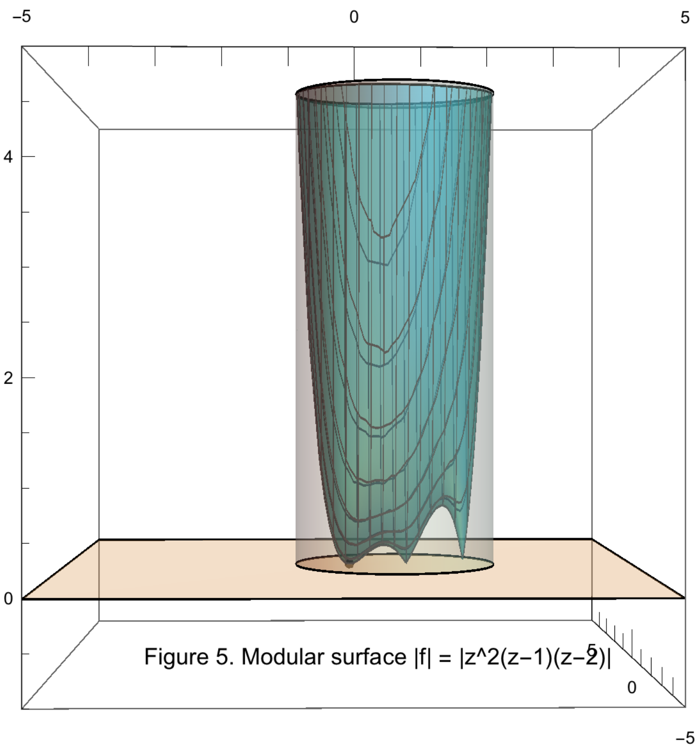

Figure 5. Modular surface |f| = |z^2(z−1)(z−2)|

(vii) Generalize the above argument to roots and critical points that are not simple.

For analytic functions, roots that are not critical points are simple (p. 347, sentence 2). This implies that roots that are critical points are not simple. If a root “a” is a critical point, it has order (n-1), “so that $f^{(n)}$ is the first nonvanishing derivative at *a*” (p. 347, ¶ 1).

Figure 5 shows a modular surface with three pits and two saddles, but this surface has two roots at zero, two more at 1 and 2, and critical points at $\frac{1}{8}\left(9-\sqrt{17}\right)$ and $z == \frac{1}{8}\left(9+\sqrt{17}\right)$. These are all simple roots and critical points. The rule P-S = 1 still holds. The pit at 0 corresponds to two roots at z = 0.

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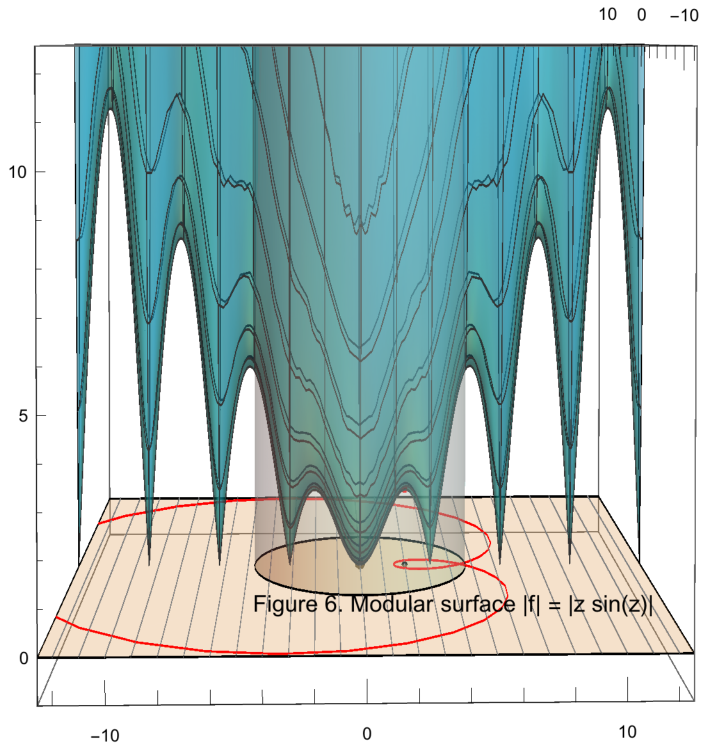

Figure 6 shows a modular surface of f = z sin(z) with three pits and two saddles within |f| = $3\pi/2$. f' = sin(z) + z cos(z). The extended surface has a root z=0 at zero and roots at $\pm n\pi$, n=0..∞, n an integer. The roots at 0 are also critical points. The rule that P-S = 1 still holds for the modular surface, so long as we choose a fixed radius for |f(p)| or a finite number of branches. The pit at 0 corresponds to z = 0 and sin(0) = 0. So z = 0 is a root with multiplicity 2. The red loop plots f(Γ), where Γ is a circle of radius 2 (not shown). The inclusion of a nonzero root in the loop suggests the existence of a critical point in addition to the one at 0 (Exercise 4, Chapter 7). By inspection, the critical point (saddle point, red) in the loop appears to be near z = 2, which appears near a zero point in a plot of f' (Figure 7).

To generalize, within |f(Γ)|, where Γ =(|z| ≤ 2), there are four roots {0, 0, π, -π} and two critical points {0, 2}. In this same disc, there are three roots not counting multiplicities and two critical points. Then, not counting multiplicities, for roots and critical points that are not simple, P - S = 1. Roots and critical points with multiplicity appear as single pits.

Figure 7. Critical points of z Sin[z]
on real line

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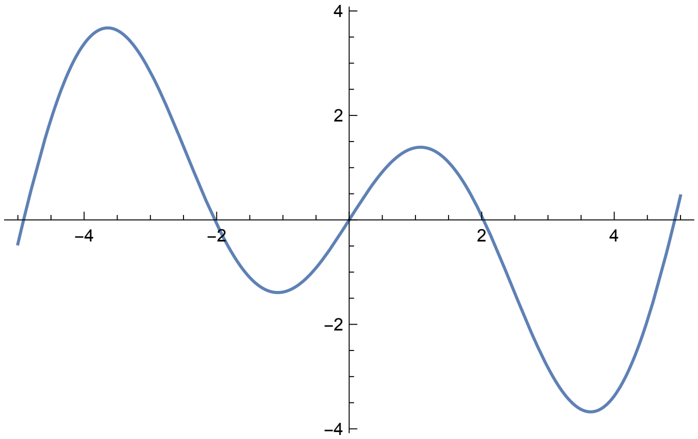

In[647]:= $\frac{1}{8}\left(9-\sqrt{17}\right)$ // N

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